Applied Functional Analysis

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Functional analysis is concerned with the study of *spaces of functions*. It offers a unifying framework and a set of tools for solving problems stemming from different areas of mathematics and knowledge. The course will present some "abstract" results (existence) and then "concrete" results (e.g. approximation, games, sampling, PDEs) solved using "abstract" tools. Examples and exercises will illustrate the important concepts and convey the feeling of the variety of applications.

1 Introduction (2h)

1.1 Vector spaces

- Subspace, linear independence, basis and dimension
- Metric spaces, open and closed subsets, convergence, complete spaces, denseness, separability

1.2 Normed linear spaces

- Definitions and examples
- Banach spaces - definition and examples
- l_p and L_p, the Holder inequality, the Minkowski inequality

2 Convexity (4h)

2.1 Generalities

- Convex functions
- Convex set, convex cone, epigraph, support functional

2.2 The Hahn-Banach theorems

- Extension of linear continuous operators
- Separation of convex sets

2.3 Applications

- The Kuhn-Tucker Theorem
- The Minimax theorem

3 Projection in Hilbert spaces (5h)

3.1 Hilbert spaces

- Inner product, orthogonality, the Schwarz inequality
- Continuous linear and bilinear operators

3.2 Orthogonal Projection

- Projection onto convex sets, projection onto a linear subspace
- Properties of the projection operator
- The fixed point theorem and the method of successive projections

3.3 Orthogonal bases

- The Gram-Schmidt procedure
- Approximation: normal equations and Gram matrices
- Complete orthonormal sequences

4 Linear transforms on Banach spaces (7h)

4.1 Sets of linear operators

- Baire's theorem
- Uniform boundedness (the Banach-Steinhaus theorem)
- The open mapping theorem
- The closed graph theorem
- Left and right invertible operators

4.2 Dual spaces

- The concept of duality, identification, examples
- Bi-duals, reflexive spaces
- Orthogonality relations in reflexive spaces
- Transposed operators, geometric interpretation

4.3 Duality in Hilbert spaces

- The Riesz representation theorem
- The theorem of Stampacchia
- The theorem of Lax-Milgram

4.4 Some applications

- Pseudo-inverse operators
- Convex conjugate functions, bi-conjugates, the theorem of Fenchel-Rockafellar

5 Elementary Spectral Theory (5h)

5.1 Compact operators

- Definition and basic properties
- Application: Approximation of continuous transforms using finite rank operators
- Compact unit balls
- Range and null space of a compact operator

5.2 Spectrum of a compact operator

- Definition and basic properties
- Spectral decomposition of self-transposed operators
- Application: Best approximation processes

6 Convolution of functions (3h)

6.1 Convolution

- Definition
- Localization and other important properties

6.2 Applications

- Approximation by convolution (regularization)
- Convolution for characteristic functions

7 Fourier Transform (9h)

7.1 Test functions and distributions

- Spaces $\mathcal{D}$, $\mathcal{S}$ and their duals
- Calculus with distributions

7.2 Classical analysis

- Definition of the Fourier transform (FT)
- Inverse formula
- Properties of the FT, Plancharel and Parseval theorems

7.3 FT in the distribution sense

- The Poisson formula
- FT of important distributions (Dirac, step, sinusoid)
- Unifying viewpoint (discrete and periodic functions or spectra, discrete FT)
- Application: sampling and reconstruction (Shannon theorem)

7.4 Application of the FT to differential equations

- Fundamental solutions
- Elliptic equations

8 Sobolev spaces (5h)

8.1 Motivation

- Weak solutions of boundary value problems

8.2 The Sobolev space $W^{1,p}$

- Important properties of functions on $W^{1,p}$
- Extension operation. Density.

8.3 The space $W_0^{1,p}$

- The inequalities of Poincaré, Poincaré-Wirtinger and Hardy.
- The dual of $W_0^{1,p}$

8.4 Examples of boundary value problems.

- Dirichlet (non) homogeneous boundary condition
- Neumamn (non) homogeneous boundary condition
- The maximum principle

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