Relationship between the optimal solutions of least squares regularized with ℓ_0-norm and constrained by k-sparsity

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Abstract. Two widely used models to find a sparse solution from a noisy underdetermined linear system are the constrained problem where the quadratic error is minimized subject to a sparsity constraint, and the regularized problem where a regularization parameter balances the minimization of both quadratic error and sparsity. However, the connections between these two problems have remained unclear so far. We provide an exhaustive description of the relationship between their globally optimal solutions. A partial equivalence between them always exists. We exhibit a sequence of critical parameters that partitions the positive axis into a certain number of intervals. For every regularization parameter inside an interval, there is a sparsity level such that the regularized problem and the constrained problem have the same global minimizers. At the values of the critical parameters, the optimal set of the regularized problem contains two optimal sets of the constrained problem. When the length of the sequence of critical parameters equals the number of all sparsity levels, both problems are quasi-completely equivalent. The critical parameters are obtained from the optimal values of the constrained problem.

Keywords: ℓ_0-regularization; k-sparsity constraint; globally optimal solutions; optimal solution analysis; parameter selection; quasi-equivalence between nonconvex problems; sparse recovery; under-determined linear systems.

1 Introduction

The recovery of sparse objects (e.g., signals, images) or representations $u \in \mathbb{R}^{\mathsf{N}}$ using a few basis vectors from few and possibly inaccurate data $d \in \mathbb{R}^{\mathsf{M}}$ is an extremely lively area of research in linear inverse problems and in compressed sensing [13, 8, 37, 14]. The most natural measure of sparsity is the counting function $\|\,.\,\|_0$, called usually the ℓ_0-norm

$$\|u\|_0 := \sharp\left\{i \in \{0, 1, \cdots, \mathsf{N}\} \ : \ u_i \neq 0\right\} , \tag{1}$$

where $\sharp S$ is the number of elements in the set S and u_i is the ith components of u. We consider a matrix (e.g., a dictionary, a measurement system) $A \in \mathbb{R}^{\mathsf{M}\times\mathsf{N}}$ with $\mathsf{M} < \mathsf{N}$ for fixed M and N.

Two desirable models to find a sparse solution are given by the following optimization problems:

- the k-sparsity constrained minimization problem where one looks for the minimum squared error at a given level of sparsity k

$$(\mathcal{C}_{\mathrm{k}}) \qquad \min_{u\in\mathbb{R}^{\mathsf{N}}} \ \|Au - d\|_2^2 \ , \quad \text{subject to} \ \ \|u\|_0 \leqslant \mathrm{k} \ , \tag{2}$$

- the $\|\cdot\|_0$-regularized problem where a positive parameter β is used to balance the minimization of both squared error and sparsity in the objective function $\mathcal{F}_\beta : \mathbb{R}^{\mathsf{N}} \to \mathbb{R}$

$$(\mathcal{R}_\beta) \qquad\qquad \mathcal{F}_\beta(u) := \|Au - d\|_2^2 + \beta\|u\|_0 \ , \quad \beta > 0 \ . \tag{3}$$

Let us evoke a few fields where problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ arise. Problem $(\mathcal{C}_{\mathrm{k}})$ involves a natural sparse coding constraint; it is a particular case of the well known best k-term approximation [11, 10]. It has been used for low-rank matrix decomposition [3], sparse inverse problems [7]. Problem $(\mathcal{R}_\beta)$ has been widely considered for subset selection [26, 5], model selection [21], variable selection [19], feature selection [29, 15], signal and image reconstruction [18, 16, 12].

Even though explored for several decades, problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ were essentially considered from a numerical standpoint. The existence of some connections between these problems seems intuitive. However, the relationship between these problems has never been studied in a systematic way.

The goal of this work is to analyze in depth the connections between the sets of global minimizers of $(\mathcal{C}_{\mathrm{k}})$ *and of* $(\mathcal{R}_\beta)$. Our theoretical results raise salient questions about the existing algorithms and can help the design of innovative numerical schemes.

As usual, we say "optimal set" or "optimal solutions" (resp., "optimal values") for *globally optimal* solutions, i.e., *global minimizers* (resp., globally optimal values) [34, 2]. For clarity we recall that:

– In problem $(\mathcal{C}_{\mathrm{k}})$ for $k \leqslant \mathsf{N}$ the constraint set of u reads as $\left\{u \in \mathbb{R}^{\mathsf{N}} \mid \|u\|_0 \leqslant \mathrm{k}\right\}$, so we have

$$\begin{aligned} &\text{optimal value} && \mathrm{c}_{\mathrm{k}} := \inf\left\{\|Au - d\|^2 \mid u \in \mathbb{R}^{\mathsf{N}} \ \text{ and } \ \|u\|_0 \leqslant \mathrm{k}\right\}, && (4)\\ &\text{optimal solutions} && \widehat{\mathrm{C}}_{\mathrm{k}} := \left\{ u \in \mathbb{R}^{\mathsf{N}} \ \text{ and } \ \|u\|_0 \leqslant \mathrm{k} \mid \ \|Au - d\|^2 = \mathrm{c}_{\mathrm{k}}\right\}. && (5) \end{aligned}$$

– In problem $(\mathcal{R}_\beta)$ for $\beta > 0$ one has

$$\begin{aligned} &\text{optimal value} && r_\beta := \inf\left\{\mathcal{F}_\beta(u) \mid u \in \mathbb{R}^{\mathsf{N}}\right\}, && (6)\\ &\text{optimal solutions} && \widehat{\mathrm{R}}_\beta := \left\{u \in \mathbb{R}^{\mathsf{N}} \mid \mathcal{F}_\beta(u) = r_\beta\right\}. && (7) \end{aligned}$$

We anticipate that for any $d \in \mathbb{R}^{\mathsf{N}}$, it holds that $\widehat{\mathrm{C}}_{\mathrm{k}} \neq \varnothing$, $\forall\, \mathrm{k} \geqslant 0$ and that $\widehat{\mathrm{R}}_\beta \neq \varnothing$, $\forall\, \beta > 0$ (Lemma 1 and Theorem 2(b), respectively). In view of these definitions, we are aimed at clarifying the relationship between the sets of global minimizers $\widehat{\mathrm{C}}_{\mathrm{k}}$ and $\widehat{\mathrm{R}}_\beta$. To this end, we adopt a blanket assumption:

H 1. *The matrix* $A \in \mathbb{R}^{\mathsf{M}\times\mathsf{N}}$ *satisfies* $\operatorname{rank}(A) = \mathsf{M} < \mathsf{N}$. *It is also assumed that* $d \neq 0$.

The quite standard Definition 1 shall be used to evaluate the extent of some properties.

Definition 1. *A property is generic on* $\mathbb{R}^{\mathsf{M}}$ *if it holds on a subset of* $\mathbb{R}^{\mathsf{M}} \setminus S$ *where* S *is closed in* $\mathbb{R}^{\mathsf{M}}$ *and its Lebesgue measure in* $\mathbb{R}^{\mathsf{M}}$ *is null.*

A generic property is clearly stronger than a property that holds only with probability one because $\mathbb{R}^{\mathsf{M}} \setminus S$ contains a dense open subset of $\mathbb{R}^{\mathsf{M}}$. Equivalently, we say that a property holds generically.

1.1 A brief tour of numerical approaches

The amount of papers dealing with problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ is huge. We present a brief summary.

Solving problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ by exhaustive search is combinatorial and NP-hard in general [36]. A major difficulty raised by these problems is the design of practical numerical schemes.

The solutions of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ are usually approximated by greedy pursuit [27], relaxation of the $\|\cdot\|_0$ penalty [36] often combined with nonconvex minimization [15, 21, 16], as well as direct optimization [26, 1]. Tropp and Wright [37] gave a comprehensive overview, mainly focused on greedy pursuits and convex relaxation. In the compressed sensing context, iterative hard thresholding has become a major technique after the convergence results of Blumensath and Davies [5], further expanded by the authors to solve $(\mathcal{C}_{\mathrm{k}})$ in [6, 7]. Convergence of several algorithms has been obtained under strong assumptions, e.g., restricted isometry property, bounds on spark(A) and sparsity of the solution [8, 6]. Certain classes of matrices, known to satisfy such conditions, are typically employed [22, 4, 9]. In the context of ill-posed inverse problems from limited data, the matrix A is fixed. Convergence of descent methods – proximal, operator splitting, and regularized Gauss-Seidel – for a wide class of problems including $(\mathcal{R}_\beta)$ and $(\mathcal{C}_{\mathrm{k}})$ was established by Attouch, Bolte and Svaiter in [1]. A promising continuous tight relaxation of problem $(\mathcal{R}_\beta)$ was recently proposed by Soubies, Blanc-Féraud and Aubert in [35].

Problem $(\mathcal{R}_\beta)$ is a particular case of a class of objectives where the counting function $\|\cdot\|_0$ is used for Markov random field models. In the inaugural work [17] Geman and Geman (1984) designed a stochastic relaxation method for labeled images that achieves global minimization asymptotically. Various approaches have been proposed to improve the convergence speed. Robini, Lachal and Magnin [31, 32] introduced the stochastic continuation approach and proved high probability for convergence to a global minimizer in finite time. They applied the method to reconstruct 3D tomographic images. Robini and Reissman [33] extended the methodology to general combinatorial objectives and gave results on the probability for global convergence with respect to the running time.

The progress in solving problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ is important. One observes that the corresponding numerical schemes share some common points. Exploring the relationship between the optimal sets of these two problems arises as a natural question.

1.2 Main contributions

This work provides a detailed description of the relationship between the sets of global minimizers (called also optimal sets) of the two nonconvex (combinatorial) problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$, given in (2) and (3), respectively. These sets, see (5) and (7), are always nonempty (Lemma 1 and Theorem 2(b)). Our main results are summarized below. The couple (A, d) satisfies H1.

- We define L as the least number so that the optimal value of $(\mathcal{C}_{\mathsf{L}})$ is null; note that $\mathsf{L} = \mathsf{M}$ generically (Proposition 5). For any $\mathrm{k} \leqslant \mathsf{L}$, any optimal solution $\widehat{u}$ of $(\mathcal{C}_{\mathrm{k}})$ is *strict* and obeys $\|\widehat{u}\|_0 = \mathrm{k}$ (Theorem 1). Problem $(\mathcal{R}_\beta)$ for all $\beta \in (0, +\infty)$ has *at most* $\mathsf{L}+1$ different sets of global minimizers which are global minimizers of $(\mathcal{C}_{\mathrm{k}})$ for $\mathrm{k} \in \{0, \ldots, \mathsf{L}\}$ (Theorem 4).
- Optimality of $(\mathcal{R}_\beta)$ can be reduced to a search over the optimal sets of $(\mathcal{C}_{\mathrm{k}})$ (Theorem 5).
- A sequence of parameter values $\{\beta_{\mathrm{k}},\, \beta_{\mathrm{k}}^{\mathrm{U}}\}_{\mathrm{k}=0}^{\mathsf{L}}$ is proposed (Definition 3) using the optimal values of problem $(\mathcal{C}_{\mathrm{k}})$. The global minimizers of $(\mathcal{C}_{\mathrm{k}})$ and of $(\mathcal{R}_\beta)$ coincide if and only if $\beta_{\mathrm{k}} < \beta < \beta_{\mathrm{k}}^{\mathrm{U}}$

(Theorem 6). However, according to the data, it can occur that $(\beta_{\mathrm{k}}, \beta_{\mathrm{k}}^{\mathrm{U}}) = \varnothing$. So we focus on $\mathrm{J} := \{\mathrm{k} \in \{0, \ldots, \mathsf{L}\} \mid \beta_{\mathrm{k}} < \beta_{\mathrm{k}}^{\mathrm{U}}\}$ which subset is always nonempty and yields $\beta_{\mathrm{J_k}}^{\mathrm{U}} = \beta_{\mathrm{J_{k-1}}}$ (Proposition 3).

- *Problem* $(\mathcal{R}_\beta)$ *for any* $\beta \in \left(\beta_{\mathrm{J_k}}, \beta_{\mathrm{J_{k-1}}}\right)$ *and problem* $(\mathcal{C}_{\mathrm{J_k}})$ *have the same global minimizers* (Theorem 7). This agreement is referred to as *partial equivalence*. For the isolated values $\beta = \beta_{\mathrm{J_k}}$, the optimal set of problem $(\mathcal{R}_\beta)$ contains the global minimizers of $(\mathcal{C}_{\mathrm{J_k}})$ and $\left(\mathcal{C}_{\mathrm{J_{k+1}}}\right)$ (Theorem 8).
- $\{\beta_{\mathrm{k}}\}_{\mathrm{k}\in\mathrm{J}}$ is the largest strictly decreasing subsequence in Definition 3 containing β_0 (Proposition 4).
- When the whole sequence $\{\beta_{\mathrm{k}}\}_{\mathrm{k}=0}^{\mathsf{L}}$ in Definition 3 is strictly decreasing, problem $(\mathcal{C}_{\mathrm{k}})$ and problem $(\mathcal{R}_\beta)$ for all $\beta \in (\beta_{\mathrm{k}}, \beta_{\mathrm{k}-1})$ have the same optimal set (Theorem 9). This case is referred to as *quasi-complete equivalence.*
- The optimal solutions of $(\mathcal{C}_{\mathrm{k}})$ and of $(\mathcal{R}_\beta)$ are generically unique (subsection 6.1).

1.3 Paper overview

Section 2 establishes necessary and sufficient conditions for global minimizers of problem $(\mathcal{R}_\beta)$ only in terms of the global minimizers of problem $(\mathcal{C}_{\mathrm{k}})$. It begins with a study of the optimal sets of problem $(\mathcal{C}_{\mathrm{k}})$. Facts on the optimal sets of problem $(\mathcal{R}_\beta)$ are taken from [30]. Section 3 is devoted to parameter values and conditions for agreement between the optimal sets of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$. The main results on the relationship between the optimal sets of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ – partial equivalence and possible quasi-complete equivalence – are established and discussed in section 4. Some facts on the optimal values of these problems are given in section 5. Uniqueness of the global minimizers of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ under additional mild conditions is discussed in section 6. The theoretical findings are illustrated using exact numerical tests for $(\mathsf{M}, \mathsf{N}) = (5, 10)$ in section 7. Conclusions and future directions are presented in section 8.

1.4 Notation

For ease of presentation, we give here all important notations used throughout the paper.

The ℓ_2-norm is denoted by $\|\,.\,\| := \|\,.\,\|_2$. Let n be a positive integer. We denote by $\mathbb{I}_{\mathrm{n}}$ and $\mathbb{I}_{\mathrm{n}}^0$ the totally and strictly ordered index sets

$$\mathbb{I}_{\mathrm{n}} := \big(\{1, \ldots, \mathrm{n}\}, <\big) \quad \text{and} \quad \mathbb{I}_{\mathrm{n}}^0 := \big(\{0, 1, \ldots, \mathrm{n}\}, <\big) \;, \tag{8}$$

where the symbol $<$ stands for the natural order of integers (the superscripts ^{0} recalls that zero is included). Thus any subset $\omega \subseteq \mathbb{I}_{\mathrm{n}}$ is also totally and strictly ordered. The support of $u \in \mathbb{R}^{\mathrm{n}}$ is $\mathrm{supp}(u) := \left\{ i \in \mathbb{I}_{\mathrm{n}}^0 \mid u[i] \neq 0 \right\}$. A vector u is said to be k-sparse if $\|u\|_0 = \sharp\,(\mathrm{supp}(u)) \leqslant \mathrm{k}$.

Remark 1. For $(\mathcal{C}_{\mathrm{k}})$ we consider also the trivial case $\mathrm{k} = 0$ because $\mathcal{F}_\beta$ always has a strict (local) minimum at $\widehat{u} = 0$ [30]. According to the value of β, $\widehat{u}$ can be global minimizer of $\mathcal{F}_\beta$. ♢

For any $\omega \subseteq \mathbb{I}_{\mathsf{N}}$, we denote by A_ω the $\mathsf{M} \times \sharp\,\omega$ submatrix of A formed from the columns of A with indexes in ω and similarly u_ω is the $\sharp\,\omega$-length restriction of $u \in \mathbb{R}^{\mathsf{N}}$ whose indexes are in ω:

$$A_\omega := \left(A_{\omega_1}, \ldots, A_{\omega_{\sharp\,\omega}}\right) \quad \text{and} \quad u_\omega := \left(u_{\omega_1}, \ldots, u_{\omega_{\sharp\,\omega}}\right)^{\mathrm{T}} \;,$$

where the superscript $^{\mathrm{T}}$ means transposed. We also set $A_\omega^{\mathrm{T}} := (A_\omega)^{\mathrm{T}}$. In view of Remark 1, we define $A_\varnothing := [\;] \in \mathbb{R}^{\mathsf{M}\times 0}$ and $\mathrm{rank}\,(A_\varnothing) := 0$ in order to handle the case $\widehat{u} = 0$. The identity operator on $\mathbb{R}^{\mathrm{n}}$

is denoted by I_{n}; the index n is omitted when clear from the context. A vector or a matrix of zeros of arbitrary dimension is denoted by 0.

The notation specific to problem $(\mathcal{C}_{\mathrm{k}})$ (resp., $(\mathcal{R}_\beta)$) is based on the letter "c" (resp., the letter "r"). Lowercase (resp., uppercase) letters stand for optimal values (resp., optimal sets). Furthermore:

- $\mathsf{L} := \min\big\{\mathrm{k} \in \mathbb{I}_{\mathsf{N}} \;\big|\; \mathrm{c}_{\mathrm{k}} = 0\big\}$ where c_{k} is the optimal value of problem $(\mathcal{C}_{\mathrm{k}})$ defined in (4).
- $\Omega_{\mathrm{k}} := \{\omega \subset \mathbb{I}_{\mathsf{N}} \mid \sharp\,\omega = \mathrm{k} = \mathrm{rank}\,(A_\omega)\}$ – introduced in (17) (section 2).
- $\widehat{\mathrm{C}} := \bigcup_{\mathrm{k}=0}^{\mathsf{L}} \widehat{\mathrm{C}}_{\mathrm{k}}$ where $\widehat{\mathrm{C}}_{\mathrm{k}}$ are he global minimizers of $(\mathcal{C}_{\mathrm{k}})$ defined in (5).
- $\widehat{\mathrm{R}} := \bigcup_{\beta>0} \widehat{\mathrm{R}}_\beta$ where $\widehat{\mathrm{R}}_\beta$ is the optimal set of $(\mathcal{R}_\beta)$ (the global minimizers of $\mathcal{F}_\beta$) given in (7).
- β_{k}, $\beta_{\mathrm{k}}^{\mathrm{U}}$ for $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0$ – critical parameter values, Definition 3 (section 3).
- J (resp. J^{E}) – all $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0$ such that $\beta_{\mathrm{k}} < \beta_{\mathrm{k}}^{\mathrm{U}}$ (resp., $\beta_{\mathrm{k}} = \beta_{\mathrm{k}}^{\mathrm{U}}$), Definition 4 (section 3).

2 Joint optimality conditions for $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$

In this section we shall derive necessary and sufficient conditions for the global minimizers of problem $(\mathcal{R}_\beta)$ only in terms of the global minimizers of problem $(\mathcal{C}_{\mathrm{k}})$. The obtained results enable us to compare the optimal sets of these problems.

2.1 Preliminaries

A constrained quadratic optimization problem Given $d \in \mathbb{R}^{\mathsf{M}}$ and $\omega \subseteq \mathbb{I}_{\mathsf{N}}$, problem $(\mathcal{P}_\omega)$ reads as:

$$(\mathcal{P}_\omega) \qquad \begin{array}{c} \displaystyle\min_{u\in\mathbb{R}^{\mathsf{N}}} \|Au-d\|^2 \quad \text{subject to} \quad u[i]=0 \quad \forall\, i \in \mathbb{I}_{\mathsf{N}}^0 \setminus \omega \\ \iff \quad \widehat{u}_\omega = \arg\min_{v\in\mathbb{R}^{\sharp\,\omega}} \|A_\omega v - d\|^2 \quad \text{and} \qquad \widehat{u}_{\mathbb{I}_{\mathsf{N}}\setminus\omega} = 0\ . \end{array} \tag{9}$$

The convex problem $(\mathcal{P}_\omega)$ is related to problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$. So problem $(\mathcal{P}_\omega)$ is a good tool for analyzing the combinatorial problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$. This is used in sections 2 and 6. We remind that for any $\omega \subset \mathbb{I}_{\mathsf{N}}^0$, the solution of problem $(\mathcal{P}_\omega)$ is a (local) minimizer of the nonconvex objective $\mathcal{F}_\beta$ in (3), see [30]. This fact is independent of the value of β.

For clarity, we recall the following definitions:

Definition 2. *For a function $f : \mathbb{R}^{\mathsf{N}} \to \mathbb{R}$ and a set $S \subseteq \mathbb{R}^{\mathsf{N}}$, $\widehat{u}$ is a* strict *(local) minimizer of the problem* $\min\{f(u) \mid u \in S\}$ *if there is a neighborhood $\mathcal{O} \subset S$ containing $\widehat{u}$ so that $f(u) > f(\widehat{u})$ for any $u \in \mathcal{O}\setminus\{\widehat{u}\}$. Further, $\widehat{u}$ is an* isolated *(local) minimizer if $\widehat{u}$ is the only minimizer in an open subset $\mathcal{O}' \subset \mathcal{O}$; see, e.g., [28].*

An isolated minimizer is always a strict minimizer.

Remark 2. For any $\omega \subset \mathbb{I}_{\mathsf{N}}$ such that $\mathrm{rank}(A_\omega) = \sharp\,\omega$, it is readily seen from (9) that the solution $\widehat{u}$ of $(\mathcal{P}_\omega)$ is an isolated minimizer. Any solution of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ is a solution of problem $(\mathcal{P}_\omega)$ for a particular ω; thus all strict minimizers discussed in this work are also isolated minimizers.

2.2 On the global minimizers of problem ($\mathcal{C}_\mathrm{k}$)

We begin with a brief study of the optimal sets of problem ($\mathcal{C}_\mathrm{k}$) that are essential to develop the paper. The proofs of all results in this subsection are given in Appendix A.1.

The list of the supports $\omega \in \mathbb{I}_\mathsf{N}$ of all k-sparse vectors in $\mathbb{R}^\mathsf{N}$ is given bellow:

$$\Sigma_\mathrm{k} := \bigcup_{\mathrm{n}=0}^{\mathrm{k}} \left\{ \omega \subset \mathbb{I}_\mathsf{N} \;\middle|\; \sharp\,\omega = \mathrm{n} \right\} . \tag{10}$$

Using this notation problem ($\mathcal{C}_\mathrm{k}$) in (2) also reads as

$$\min_{u\in\mathbb{R}^\mathsf{N}} \|Au - d\|^2 \ , \quad \text{subject to} \quad \mathrm{supp}(u) \in \Sigma_\mathrm{k} \ . \tag{11}$$

The corresponding optimal set $\widehat{\mathrm{C}}_\mathrm{k}$ given in (5) is

$$\widehat{\mathrm{C}}_\mathrm{k} = \left\{ u \in \mathbb{R}^\mathsf{N},\ \mathrm{supp}(u) \in \Sigma_\mathrm{k} \;\middle|\; \|Au - d\|^2 = \mathrm{c}_\mathrm{k} \right\} . \tag{12}$$

It is straightforward that if $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}$ then $\widehat{u}$ solves ($\mathcal{P}_\omega$) for $\omega := \mathrm{supp}(\widehat{u})$.

A central question is to know whether problem ($\mathcal{C}_\mathrm{k}$) admits an optimal solution.

Lemma 1. *For any* $\mathrm{k} \in \mathbb{I}_\mathsf{N}^0$ *problem* ($\mathcal{C}_\mathrm{k}$) *has a global minimizer, i.e.,* $\widehat{\mathrm{C}}_\mathrm{k} \neq \varnothing$.

Using ($\mathcal{P}_\omega$) in (9) and (11), the optimal value c_k of ($\mathcal{C}_\mathrm{k}$) is

$$\mathrm{c}_\mathrm{k} = \min \left\{ \|A\widetilde{u} - d\|^2 \text{ where } \widetilde{u} \in \mathbb{R}^\mathsf{N} \text{ solves } (\mathcal{P}_\omega) \;\middle|\; \omega \in \Sigma_\mathrm{k} \right\} . \tag{13}$$

At this point, we need the following simple lemma.

Lemma 2. *Let H1 hold. Then* $\mathrm{c}_0 = \|d\|^2 > 0$ *and* $\{\mathrm{c}_\mathrm{k}\}_{\mathrm{k}\geqslant 0}$ *is decreasing with* $\mathrm{c}_\mathrm{k} = 0 \ \ \forall\, \mathrm{k} \geqslant \mathsf{M}$.

The next simple lemma has a pivotal role in this work.

Lemma 3. *Let H1 hold. For* $\mathrm{k} \in \mathbb{I}_\mathsf{M}$, *assume that* ($\mathcal{C}_\mathrm{k}$) *has an optimal solution* $\widehat{u}$ *obeying*

$$\|\widehat{u}\|_0 = \mathrm{k} - \mathrm{n} \quad \textit{for} \quad \mathrm{n} \geqslant 1 \ . \tag{14}$$

Then $A\widehat{u} = d$. *Furthermore,* $\quad \mathrm{c}_\mathrm{m} = 0 \ \text{ and } \ \widehat{u} \in \widehat{\mathrm{C}}_\mathrm{m} \quad \forall\, \mathrm{m} \geqslant \mathrm{k} - \mathrm{n}$.

Based on Lemma 3 and assuming that H1 holds, we introduce the constant

$$\boxed{\mathsf{L} := \min\left\{\mathrm{k} \in \mathbb{I}_\mathsf{N} \mid \mathrm{c}_\mathrm{k} = 0\right\} .} \tag{15}$$

L is uniquely defined since $\{\mathrm{c}_\mathrm{k}\}$ is decreasing with $\mathsf{L} \leqslant \mathsf{M}$ (Lemma 2). We emphasize that L depends only on d (Lemma 5(a)) and that $\mathsf{L} = \mathsf{M}$ generically (Proposition 5(b) and Remark 9).

Example 1. One has $\mathsf{L} \leqslant \mathsf{M} - 1$ if $d = Au$ for $\|u\|_0 \leqslant \mathsf{M} - 1$. Then d belongs to a subspace of $\mathbb{R}^\mathsf{M}$ of dimension $\|u\|_0$ which has null Lebesgue measure in $\mathbb{R}^\mathsf{M}$. Usual data range on the whole $\mathbb{R}^\mathsf{M}$ and $\mathsf{L} = \mathsf{M}$.

Theorem 1. *Let H1 hold and* L *be as in* (15). *One has:*

(a) *If* $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0$, *then*

$$\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}} \quad \Longrightarrow \quad \|\widehat{u}\|_0 = \mathrm{k} = \mathrm{rank}\,(A_{\widehat{\sigma}}) \quad \text{for} \quad \widehat{\sigma} := \mathrm{supp}(\widehat{u}) \tag{16}$$

and the global minimizer $\widehat{u}$ *of problem* $(\mathcal{C}_{\mathrm{k}})$ *is strict.*

(b) *If* $\mathrm{k} \geqslant \mathsf{L}+1$ *then* $\widehat{\mathrm{C}}_{\mathsf{L}} \subset \widehat{\mathrm{C}}_{\mathrm{k}}$.

Observe that $\widehat{u}$ in (16) is a strict solution of problem $(\mathcal{C}_{\mathrm{k}})$ since $\widehat{u}$ solves problem $(\mathcal{P}_{\widehat{\sigma}})$ in (9) for $\widehat{\sigma} := \mathrm{supp}(\widehat{u})$ where $\mathrm{rank}\,(A_{\widehat{\sigma}}) = \sharp\,\widehat{\sigma}$ (Lemma 10). Further, the graph $\{(\mathrm{k}, \mathrm{c}_{\mathrm{k}}) \mid \mathrm{k} \leqslant \mathsf{L}\}$ is Pareto optimal [23, Definition 1]: for any $\mathrm{k} \leqslant \mathsf{L}$, there is no $u \notin \widehat{\mathrm{C}}_{\mathrm{k}}$ that can decrease both $\|Au - d\|^2$ and $\|u\|_0$.

Theorem 1(a) gives necessary conditions for an optimal solution of $(\mathcal{C}_{\mathrm{k}})$ for $\mathrm{k} \leqslant \mathsf{L}$.

The optimal solutions of $(\mathcal{C}_{\mathrm{k}})$ for $\mathrm{k} \leqslant \mathsf{L}$ are strict minimizers.

The algorithm aimed at solving $(\mathcal{C}_{\mathrm{k}})$ proposed in [5] was shown in [5, Lemma 6] to produce, under certain conditions, solutions that fulfill this necessary condition.

Example 2. Let I_2 be the 2×2 identity matrix and set $\mathbb{1}_2 := (1,1)^{\mathrm{T}}$.

- Consider that $A = (I_2, \mathbb{1}_2)$ and $d = \mathbb{1}_2$. There is an optimal solution $\widehat{u} = (0,0,1)^{\mathrm{T}}$ with $\mathrm{c}_1 = 0$, hence $\mathsf{L} = 1$. Further, $\widehat{u} = (1,1,0)^{\mathrm{T}}$ is a strict minimizer of $(\mathcal{C}_2)$ because $\mathrm{rank}\left(A_{\mathrm{supp}(\widehat{u})}\right) = 2$. One has $\|\widehat{u}\|_0 = 3$ for a continuum of optimal solutions of the form $\widehat{u} = (x, x, 1-x)^{\mathrm{T}}$ where $x \in \mathbb{R} \setminus \{0,1\}$.
- Let now $A = (I_2, I_2)$ and $d = (1,0)^{\mathrm{T}}$. The optimal solutions of $(\mathcal{C}_1)$ are $(1,0,0,0)^{\mathrm{T}}$ and $(0,0,1,0)^{\mathrm{T}}$. One has $\mathrm{c}_1 = 0$ and $\mathsf{L} = 1$. For $\mathrm{k} \geqslant 2$ all *other* optimal solutions are nonstrict and have the form $\widehat{u} = (x, y, 1-x, -y)^{\mathrm{T}}$, $x \in \mathbb{R} \setminus \{0,1\}$. If $y = 0$ then $\|\widehat{u}\|_0 = 2$ and otherwise $\|\widehat{u}\|_0 = 4$. $\diamondsuit$

The optimal solutions of $(\mathcal{C}_{\mathrm{k}})$ for $\mathrm{k} > \mathsf{L}$ have ℓ_0-norms in $\{\mathsf{L}, \ldots, \mathrm{k}\}$ and they can be nonstrict.

Remark 3. [*On Assumption* H1] This usual assumption ensures that A is rich enough to represent any $d \in \mathbb{R}^{\mathsf{M}}$. It is also needed to prove the pivotal Lemma 3 and to define L in (15). $\diamondsuit$

A direct and useful consequence of Theorem 1(a) is stated below.

Corollary 1. *Let H1 hold. Then* $\widehat{\mathrm{C}}_{\mathrm{k}} \cap \widehat{\mathrm{C}}_{\mathrm{n}} = \varnothing$ *for all* $(\mathrm{k}, \mathrm{n}) \in (\mathbb{I}_{\mathsf{L}}^0)^2$ *such that* $\mathrm{k} \neq \mathrm{n}$.

If $\widehat{u}$ solves optimally $(\mathcal{C}_{\mathrm{k}})$ for $\mathrm{k} \leqslant \mathsf{L}$, then $\widehat{u}$ is not an optimal solution of $(\mathcal{C}_{\mathrm{n}})$ for $\mathrm{n} \leqslant \mathsf{L}$, $\mathrm{n} \neq \mathrm{k}$.

By Theorem 1(a), many subsets in $\{\Sigma_{\mathrm{k}}\}_{\mathrm{k}=0}^{\mathsf{N}}$ are not the supports of optimal solutions of $(\mathcal{C}_{\mathrm{k}})$. Accordingly, we focus only on the subsets $\omega \subset \mathbb{I}_{\mathsf{N}}^0$ with exactly k entries such that $\mathrm{rank}\,(A_\omega) = \mathrm{k}$:

$$\Omega_{\mathrm{k}} := \left\{ \ \omega \subset \mathbb{I}_{\mathsf{N}} \ \mid \ \sharp\,\omega = \mathrm{k} = \mathrm{rank}\,(A_\omega) \ \right\}. \tag{17}$$

Remark 4. From H1 and Theorem 1(a), the optimal value of problem $(\mathcal{C}_{\mathrm{k}})$ for any $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0$ obeys

$$\mathrm{c}_{\mathrm{k}} = \min \left\{ \ \|A\widetilde{u} - d\|^2 \text{ where } \widetilde{u} \in \mathbb{R}^{\mathsf{N}} \ \text{ solves } (\mathcal{P}_\omega) \ \middle| \ \omega \in \Omega_{\mathrm{k}} \ \right\} \tag{18}$$

and the corresponding set of optimal solutions satisfies

$$\widehat{\mathrm{C}}_{\mathrm{k}} = \left\{ u \in \mathbb{R}^{\mathsf{N}}, \ \mathrm{supp}(u) \in \Omega_{\mathrm{k}} \ \middle| \ \|Au - d\|^2 = \mathrm{c}_{\mathrm{k}} \ \right\}. \tag{19}$$

Let $\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}$; for $\widehat{\sigma} := \mathrm{supp}(\widehat{u})$ one has $\widehat{\sigma} \in \Omega_{\mathrm{k}}$ and $\widehat{u}$ solves $(\mathcal{P}_{\widehat{\sigma}})$, hence $\widehat{u}$ is an isolated minimizer (Remark 2). Denoting by $\Pi_{\widehat{\sigma}}$ the orthogonal projector onto $\mathrm{range}\,(A_{\widehat{\sigma}})$, see e.g. [25], one has

$$\mathrm{c}_{\mathrm{k}} = d^{\mathrm{T}}(I - \Pi_{\widehat{\sigma}})\,d \quad \text{where} \quad \Pi_{\widehat{\sigma}} = A_{\widehat{\sigma}}\,(A_{\widehat{\sigma}}^{\mathrm{T}} A_{\widehat{\sigma}})^{-1}\,A_{\widehat{\sigma}}^{\mathrm{T}}\,. \tag{20}$$

From (18) and (20), the optimal value c_{k} satisfies also $\mathrm{c}_{\mathrm{k}} = \|d\|^2 - \max\left\{d^{\mathrm{T}}\,\Pi_\omega\,d \ \middle| \ \omega \in \Omega_{\mathrm{k}}\right\}$. $\diamondsuit$

The fact that $\sharp\,\Omega_{\mathrm{k}} \ll \sharp\,\Sigma_{\mathrm{k}}$ might be useful.

2.3 Necessary and sufficient conditions

Problem $(\mathcal{R}_\beta)$ in (3) is equivalently given by the global minimization of

$$\mathcal{F}_\beta(u) = \|Au - d\|^2 + \beta\,\sharp\,\mathrm{supp}(u)\ .$$

We give two known results on its optimal sets (Theorems 2 and 3) needed in what follows. They hold for any $A \in \mathbb{R}^{\mathsf{M}\times\mathsf{N}}$ with $\mathsf{M} < \mathsf{N}$.

Theorem 2 ([30] Theorem 4.4)**.** *Let $\mathcal{F}_\beta$ read as in* (3)*. The following statements hold:*

(a) *$\mathcal{F}_\beta$ always has a global minimizer, i.e., $\widehat{\mathrm{R}}_\beta \neq \varnothing$ for any $\beta > 0$ and any $d \in \mathbb{R}^{\mathsf{M}}$.*

(b) *If $\widehat{u}$ is a global minimizer of $\mathcal{F}_\beta$, then $\widehat{u}$ is a strict minimizer.*

The global minimizers of $\mathcal{F}_\beta$ are strict. The strict (local) minimizers of $\mathcal{F}_\beta$ are characterized next.

Theorem 3 ([30], Theorem 3.2)**.** *A point $\widehat{u} \in \mathbb{R}^{\mathsf{N}}$ is a* strict *(local) minimizer of $\mathcal{F}_\beta$ if and only if* $\mathrm{rank}\,(A_{\widehat{\sigma}}) = \sharp\,\widehat{\sigma}$ *where $\widehat{\sigma} := \mathrm{supp}(\widehat{u})$.*

The global minimizers $\widehat{u}$ of $(\mathcal{C}_{\mathrm{k}})$ for $k \leqslant \mathsf{L}$ and those of $\mathcal{F}_\beta$ are strict and moreover isolated: they solve problem $(\mathcal{P}_{\widehat{\sigma}})$ in (9) for $\widehat{\sigma} := \mathrm{supp}(\widehat{u})$ such that $\mathrm{rank}(A_{\widehat{\sigma}}) = \sharp\,\widehat{\sigma}$ (Remark 2).

The proofs of the statements below (except for Theorem 5) are given in Appendix A.2.

Proposition 1 relates the optimal sets and the optimal values of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$.

Proposition 1. *Let H1 hold. The following statements hold:*

(a) *For any $\mathrm{k} \in \mathbb{I}^0_{\mathsf{L}}$ it holds that*

$$\mathcal{F}_\beta(\widehat{u}) = \mathrm{c}_{\mathrm{k}} + \beta\,\mathrm{k} \quad \forall\, \widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}\ .$$

(b) $\widehat{u} \in \widehat{\mathrm{R}}_\beta \quad \Longrightarrow \quad \widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}$ *where* $\mathrm{k} := \|\widehat{u}\|_0 \in \mathbb{I}^0_{\mathsf{L}}$.

(c) $\widehat{u} \in \widehat{\mathrm{R}}_\beta \quad \Longrightarrow \quad \widehat{\mathrm{C}}_{\mathrm{k}} \subseteq \widehat{\mathrm{R}}_\beta$ *for* $\mathrm{k} := \|\widehat{u}\|_0 \in \mathbb{I}^0_{\mathsf{L}}$.

By (c), the global minimizers of $\mathcal{F}_\beta$ are composed of some optimal sets $\widehat{\mathrm{C}}_{\mathrm{k}}$ *only* for $\mathrm{k} \leqslant \mathsf{L}$.

The statements that follow are given in terms of the optimal sets $\widehat{\mathrm{C}}_{\mathrm{k}}$ and $\widehat{\mathrm{R}}_\beta$, as defined in (5) and (7), respectively. The claim in the next Lemma 4 is usually false for ordinary subsets C and R.

Lemma 4. *Let H1 hold. For any $\beta > 0$ and for any $\mathrm{k} \in \mathbb{I}^0_{\mathsf{L}}$ one has*

$$\widehat{\mathrm{C}}_{\mathrm{k}} \not\subseteq \widehat{\mathrm{R}}_\beta \quad \Longleftrightarrow \quad \widehat{\mathrm{C}}_{\mathrm{k}} \cap \widehat{\mathrm{R}}_\beta = \varnothing\ .$$

We denote by $\widehat{\mathrm{C}}$ the collection of all optimal solutions of problems $(\mathcal{C}_{\mathrm{k}})$ for $\mathrm{k} \in \mathbb{I}^0_{\mathsf{L}}$ and likewise, by $\widehat{\mathrm{R}}$ – the set of all global minimizers of $\mathcal{F}_\beta$ for all $\beta > 0$:

$$\boxed{\widehat{\mathrm{C}} := \bigcup_{\mathrm{k}=0}^{\mathsf{L}} \widehat{\mathrm{C}}_{\mathrm{k}} \quad \text{and} \quad \widehat{\mathrm{R}} := \bigcup_{\beta>0} \widehat{\mathrm{R}}_\beta\ .} \tag{21}$$

With this notation, Theorem 4 is a direct consequence of Proposition 1(b).

Theorem 4. *Let H1 hold. Then* $\widehat{\mathrm{R}} \subset \widehat{\mathrm{C}}$.

Theorem 4 shows that when β ranges on $(0,+\infty)$, $\mathcal{F}_\beta$ can have *at most* $\mathsf{L}+1$ different sets of global minimizers which are optimal solutions of $(\mathcal{C}_{\mathrm{k}})$ for $\mathrm{k}\in\{0,\dots,\mathsf{L}\}$.

In view of Proposition 1(c) and Theorem 4, each global minimizer of $(\mathcal{R}_\beta)$ can be composed out of the optimal sets of several problems of the form of $(\mathcal{C}_{\mathrm{k}})$ or it can be equal to the optimal set of exactly one problem $(\mathcal{C}_{\mathrm{k}})$. This is made explicit in the following remark:

Remark 5. Let $\beta>0$ and $\mathrm{k}\in\mathbb{I}_{\mathsf{L}}^0$. Since $\widehat{\mathrm{R}}_\beta$ is the set of the global minimizers of $\mathcal{F}_\beta$, one has

$$\begin{aligned}\widehat{\mathrm{C}}_{\mathrm{k}}\subseteq\widehat{\mathrm{R}}_\beta \quad&\Longleftrightarrow\quad \mathcal{F}_\beta(u)\geqslant\mathcal{F}_\beta(\widehat{u})\quad\forall\,\widehat{u}\in\widehat{\mathrm{C}}_{\mathrm{k}}\quad\forall\,u\in\mathbb{R}^{\mathsf{N}}\ ;\\ \widehat{\mathrm{C}}_{\mathrm{k}}=\widehat{\mathrm{R}}_\beta \quad&\Longleftrightarrow\quad \mathcal{F}_\beta(u)>\mathcal{F}_\beta(\widehat{u})\quad\forall\,\widehat{u}\in\widehat{\mathrm{C}}_{\mathrm{k}}\quad\forall\,u\in\mathbb{R}^{\mathsf{N}}\setminus\widehat{\mathrm{C}}_{\mathrm{k}}\ .\end{aligned}$$

The next theorem provides the basic tool to compare the optimal sets of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$.

Theorem 5. *Let H1 hold and let $\beta>0$. For any $\mathrm{k}\in\mathbb{I}_{\mathsf{L}}^0$ the following holds:*

(a) $\widehat{\mathrm{C}}_{\mathrm{k}}\subseteq\widehat{\mathrm{R}}_\beta$ *if and only if*

$$\mathcal{F}_\beta(\overline{u})-\mathcal{F}_\beta(\widehat{u})\geqslant 0\quad\forall\,\widehat{u}\in\widehat{\mathrm{C}}_{\mathrm{k}}\qquad\forall\,\overline{u}\in\widehat{\mathrm{C}}\ ;\tag{22}$$

(b) $\widehat{\mathrm{C}}_{\mathrm{k}}=\widehat{\mathrm{R}}_\beta$ *if and only if*

$$\mathcal{F}_\beta(\overline{u})-\mathcal{F}_\beta(\widehat{u})>0\qquad\forall\,\widehat{u}\in\widehat{\mathrm{C}}_{\mathrm{k}}\quad\forall\,\overline{u}\in\widehat{\mathrm{C}}\setminus\widehat{\mathrm{C}}_{\mathrm{k}}\ .\tag{23}$$

Proof. From Remark 5 it is straightforward that

$$\widehat{\mathrm{C}}_{\mathrm{k}}\subseteq\widehat{\mathrm{R}}_\beta\ \ (\text{resp., }\ \widehat{\mathrm{C}}_{\mathrm{k}}=\widehat{\mathrm{R}}_\beta)\ \Longrightarrow\ \mathcal{F}_\beta(\overline{u})-\mathcal{F}_\beta(\widehat{u})\geqslant 0\ \ \forall\,\overline{u}\in\widehat{\mathrm{C}}\ \ (\text{resp., }\ >0,\ \ \forall\,\overline{u}\in\widehat{\mathrm{C}}\setminus\widehat{\mathrm{C}}_{\mathrm{k}})\quad\forall\,\widehat{u}\in\widehat{\mathrm{C}}_{\mathrm{k}}\ .$$

The rest of the proof is by contraposition.

(a) Assume that $\widehat{\mathrm{C}}_{\mathrm{k}}\not\subseteq\widehat{\mathrm{R}}_\beta$. Then $\widehat{\mathrm{C}}_{\mathrm{k}}\cap\widehat{\mathrm{R}}_\beta=\varnothing$ by Lemma 4. Since $\widehat{\mathrm{R}}_\beta\neq\varnothing$ (Theorem 2(a)), Proposition 1(c) entails that there exists $\mathrm{n}\in\mathbb{I}_{\mathsf{L}}^0\setminus\{\mathrm{k}\}$ such that $\widehat{\mathrm{C}}_{\mathrm{n}}\subseteq\widehat{\mathrm{R}}_\beta$. It follows that

$$\mathcal{F}_\beta(\widehat{u})>\mathcal{F}_\beta(\overline{u})\quad\forall\,\widehat{u}\in\widehat{\mathrm{C}}_{\mathrm{k}}\quad\forall\,\overline{u}\in\widehat{\mathrm{C}}_{\mathrm{n}}\ ,\tag{24}$$

a contradiction to (22).

(b) Let $\widehat{\mathrm{C}}_{\mathrm{k}}\neq\widehat{\mathrm{R}}_\beta$. This, together with $\widehat{\mathrm{R}}_\beta\neq\varnothing$, implies that there is $\overline{u}\in\widehat{\mathrm{R}}_\beta$ such that $\overline{u}\not\in\widehat{\mathrm{C}}_{\mathrm{k}}$. Proposition 1(c) shows that $\widehat{\mathrm{C}}_{\mathrm{n}}\subseteq\widehat{\mathrm{R}}_\beta$ for $\mathrm{n}:=\|\overline{u}\|_0\in\mathbb{I}_{\mathsf{L}}^0$ where $\mathrm{n}\neq\mathrm{k}$ (Theorem 1(a)). Therefore

$$\mathcal{F}_\beta(\widehat{u})\geqslant\mathcal{F}_\beta(\overline{u})\quad\forall\,\widehat{u}\in\widehat{\mathrm{C}}_{\mathrm{k}}\quad\forall\,\overline{u}\in\widehat{\mathrm{C}}_{\mathrm{n}}\ ,$$

which contradicts (23). $\square$

Theorem 5 is the key to finding the links between the optimal sets of $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$: it provides necessary and sufficient conditions for optimality of $(\mathcal{R}_\beta)$ only in terms of the optimal sets of $\{(\mathcal{C}_{\mathrm{k}})\}_{\mathrm{k}=0}^{\mathsf{L}}$.

A simple useful result is stated next.

Lemma 5. *Let H1 hold. Let $(\mathrm{k},\ \mathrm{k}+\mathrm{p})\in\left(\mathbb{I}_{\mathsf{L}}^0\right)^2$ for $\mathrm{p}\geqslant 1$. The following implications hold:*

$$\widehat{\mathrm{C}}_{\mathrm{k}}\subseteq\widehat{\mathrm{R}}_\beta\ \left(\textit{resp., }\ \widehat{\mathrm{C}}_{\mathrm{k}}=\widehat{\mathrm{R}}_\beta\right)\ \text{ and }\ \widehat{\mathrm{C}}_{\mathrm{k+p}}=\widehat{\mathrm{R}}_{\beta'}\left(\textit{resp., }\ \widehat{\mathrm{C}}_{\mathrm{k+p}}\subseteq\widehat{\mathrm{R}}_{\beta'}\right)\quad\Longrightarrow\quad\beta'<\beta\ .$$

This lemma confirms the intuition that when β increases on $(0,+\infty)$, the optimal sets $\widehat{\mathrm{R}}_\beta$ are given by a subsequence of $\left\{\widehat{\mathrm{C}}_{\mathrm{k}}\right\}$ with decreasing indexes.

3 Parameter values for equality between optimal sets

The links between the optimal solutions of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ are driven by the values of k and of β. This section is devoted to parameter selection.

3.1 The entire list of critical parameter values

Based on the optimal values $\{\mathrm{c_k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ of problems $(\mathcal{C}_{\mathrm{k}})$'s, see (20), for each $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0$ we give explicit formulae for the lower and the upper bounds of β that can enable an agreement between the optimal sets $\widehat{\mathrm{C}}_{\mathrm{k}}$ and $\widehat{\mathrm{R}}_\beta$.

Definition 3. (Critical parameter values) *Let* L *be as in* (15). *The parameters* $(\beta_{\mathrm{k}}, \beta_{\mathrm{k}}^{\mathrm{U}})$ *are defined by*

$$\beta_{\mathrm{k}} \;:=\; \max\left\{\frac{\mathrm{c_k} - \mathrm{c_{k+n}}}{\mathrm{n}} \;\mid\; \mathrm{n} \in \{1,\ldots,\mathsf{L}-\mathrm{k}\}\right\} \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}-1}^0 \quad \text{and} \quad \beta_{\mathsf{L}} := 0\;, \tag{25}$$

$$\beta_{\mathrm{k}}^{\mathrm{U}} \;:=\; \min\left\{\frac{\mathrm{c_{k-n}} - \mathrm{c_k}}{\mathrm{n}} \;\mid\; \mathrm{n} \in \{1,\ldots,\mathrm{k}\}\right\} \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}} \quad \text{and} \quad \beta_0^{\mathrm{U}} := +\infty\;. \tag{26}$$

The superscript $^{\mathrm{U}}$ in (26) suggests that $\beta_{\mathrm{k}}^{\mathrm{U}}$ can be an upper bound.

Remark 6. Since $d \neq 0$, this definition indicates that $\beta_{\mathsf{L}} = 0 < \beta_{\mathsf{L}}^{\mathrm{U}} = \min_{\mathrm{n}=1}^{\mathsf{L}} \frac{\mathrm{c}_{\mathsf{L}-\mathrm{n}}}{\mathrm{n}}$ and that $\beta_0 < \beta_0^{\mathrm{U}}$ because β_0 is finite. The cases where $\beta_{\mathrm{k}} < \beta_{\mathrm{k}}^{\mathrm{U}}$ will be of particular interest, as seen in section 4 (Theorems 7 and 9). However, this inequality can fail (see Discussion on Theorem 9). A simplification of these parameters is derived in Proposition 3. Some insight can be gained from the numerical tests in section 7. ♢

In view of Definition 3, the intuition suggests that the set $\{\mathrm{k} \mid \beta_{\mathrm{k}} = \beta_{\mathrm{k}}^{\mathrm{U}}\}$ should be "small".

Proposition 2. *Let H1 hold and* $\{\beta_{\mathrm{k}}, \beta_{\mathrm{k}}^{\mathrm{U}}\}_{\mathrm{k}=0}^{\mathsf{L}}$ *be as in Definition 3. There exists a finite union of vector subspaces of dimension* $\leqslant \mathsf{M} - 1$, *denoted by* S, *such that*

$$d \in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{S} \quad\Longrightarrow\quad \beta_{\mathrm{k}} \neq \beta_{\mathrm{k}}^{\mathrm{U}} \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0\;.$$

Thus $\mathbb{R}^{\mathsf{M}} \setminus \mathrm{S}$ contains a dense open subset of $\mathbb{R}^{\mathsf{M}}$. The proof is given in Appendix B.1.

Data generically live in $\mathbb{R}^{\mathsf{M}} \setminus \mathrm{S}$. So $\beta_{\mathrm{k}} \neq \beta_{\mathrm{k}}^{\mathrm{U}} \;\; \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0$ in Definition 3 holds generically.

3.2 Conditions for agreement between the optimal sets of $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$

Theorem 6 relates Theorem 5 and Definition 3. It provides a general mechanism for comparing the optimal sets of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$.

Theorem 6. *Let H1 hold. Then* $\forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0$ *it holds that*

$$\text{(a)} \qquad \widehat{\mathrm{C}}_{\mathrm{k}} \subseteq \widehat{\mathrm{R}}_\beta \qquad \textit{if and only if} \qquad \begin{cases} \beta_0 \leqslant \beta < \beta_0^{\mathrm{U}} & \text{for} \quad \mathrm{k} = 0\;; \\ \beta_{\mathrm{k}} \leqslant \beta \leqslant \beta_{\mathrm{k}}^{\mathrm{U}} & \text{for} \quad \mathrm{k} \in \{1,\ldots,\mathsf{L}-1\}\;; \\ \beta_{\mathsf{L}} < \beta \leqslant \beta_{\mathsf{L}}^{\mathrm{U}} & \text{for} \quad \mathrm{k} = \mathsf{L}\;. \end{cases}$$

$$\text{(b)} \qquad \widehat{\mathrm{C}}_{\mathrm{k}} = \widehat{\mathrm{R}}_\beta \qquad \textit{if and only if} \qquad \beta_{\mathrm{k}} < \beta < \beta_{\mathrm{k}}^{\mathrm{U}}\;.$$

Proof. The following equalities come from Proposition 1(a):
– If $\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}$ for $\mathrm{k} \in \mathbb{I}^0_{\mathsf{L}-1}$, then

$$\forall\, \mathrm{n} \in \mathbb{I}_{\mathsf{L}-\mathrm{k}} \quad \forall\, \overline{u} \in \widehat{\mathrm{C}}_{\mathrm{k+n}} \qquad \mathcal{F}_\beta(\overline{u}) - \mathcal{F}_\beta(\widehat{u}) = \mathrm{c}_{\mathrm{k+n}} - \mathrm{c}_{\mathrm{k}} + \mathrm{n}\beta = \mathrm{n}\left(\beta - \frac{\mathrm{c}_{\mathrm{k}} - \mathrm{c}_{\mathrm{k+n}}}{\mathrm{n}}\right) . \tag{27}$$

– If $\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}$ for $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}$, then

$$\forall\, \mathrm{n} \in \mathbb{I}_{\mathrm{k}} \quad \forall\, \overline{u} \in \widehat{\mathrm{C}}_{\mathrm{k-n}} \qquad \mathcal{F}_\beta(\overline{u}) - \mathcal{F}_\beta(\widehat{u}) = \mathrm{c}_{\mathrm{k-n}} - \mathrm{c}_{\mathrm{k}} - \mathrm{n}\beta = \mathrm{n}\left(\frac{\mathrm{c}_{\mathrm{k-n}} - \mathrm{c}_{\mathrm{k}}}{\mathrm{n}} - \beta\right) . \tag{28}$$

(a) Using (27) together with (25) in Definition 3, for any $\mathrm{k} \in \mathbb{I}^0_{\mathsf{L}-1}$ one has

$$\mathcal{F}_\beta(\overline{u}) \geqslant \mathcal{F}_\beta(\widehat{u}) \;\; \forall\, \widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}} \;\; \forall\, \mathrm{n} \in \mathbb{I}_{\mathsf{L}-\mathrm{k}} \;\; \forall\, \overline{u} \in \widehat{\mathrm{C}}_{\mathrm{k+n}} \iff \beta \geqslant \frac{\mathrm{c}_{\mathrm{k}} - \mathrm{c}_{\mathrm{k+n}}}{\mathrm{n}} \;\; \forall\, \mathrm{n} \in \mathbb{I}_{\mathsf{L}-\mathrm{k}} \iff \beta \geqslant \beta_{\mathrm{k}} \, .$$

Using (28) together with (26) in Definition 3, for any $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}$ one has

$$\mathcal{F}_\beta(\overline{u}) \geqslant \mathcal{F}_\beta(\widehat{u}) \;\; \forall\, \widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}} \;\; \forall\, \mathrm{n} \in \mathbb{I}_{\mathrm{k}} \;\; \forall\, \overline{u} \in \widehat{\mathrm{C}}_{\mathrm{k-n}} \iff \beta \leqslant \frac{\mathrm{c}_{\mathrm{k-n}} - \mathrm{c}_{\mathrm{k}}}{\mathrm{n}} \;\; \forall\, \mathrm{n} \in \mathbb{I}_{\mathrm{k}} \iff \beta \leqslant \beta^{\mathrm{U}}_{\mathrm{k}} \, .$$

For $\mathrm{k} = 0$ (resp., for $\mathrm{k} = \mathsf{L}$), $\beta \geqslant \beta_0$ (resp., $\beta \leqslant \beta^{\mathrm{U}}_{\mathsf{L}}$) is equivalent to $\beta_0 \leqslant \beta < +\infty =: \beta^{\mathrm{U}}_0$ (resp., $\beta_{\mathsf{L}} := 0 < \beta \leqslant \beta^{\mathrm{U}}_{\mathsf{L}}$). Combining the obtained results, one has

$$\mathcal{F}_\beta(\overline{u}) - \mathcal{F}_\beta(\widehat{u}) \geqslant 0, \quad \forall\, \widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}, \quad \forall\, \overline{u} \in \widehat{\mathrm{C}}$$

if and only if $\beta_{\mathrm{k}} \leqslant \beta \leqslant \beta^{\mathrm{U}}_{\mathrm{k}}$ for $\mathrm{k} \in \mathbb{I}_{\mathsf{L}-1}$, $\beta_0 \leqslant \beta \leqslant \beta^{\mathrm{U}}_0$ (resp., $\beta_{\mathsf{L}} < \beta \leqslant \beta^{\mathrm{U}}_{\mathsf{L}}$) for $\mathrm{k} = 0$ (resp., for $\mathrm{k} = \mathsf{L}$). Applying Theorem 5(a) proves statement (a).

(b) The proof of (b) follows the same recipe as above (nonstrict inequalities are replaced by strict inequalities) and the conclusion is obtained using Theorem 5(b). □

The proof of Theorem 6 reveals how the critical parameters in Definition 3 were defined.

3.3 The effective parameters values

Since the global minimizers of $\mathcal{F}_\beta$ are always in $\widehat{\mathrm{C}}$ (Theorem 4), we are interested in the indexes k for which there exist values of β such that $\mathcal{F}_\beta$ has global minimizers containing components of $\widehat{\mathrm{C}}_{\mathrm{k}}$. Their set, referred to as effective parameter set, is obtained from Theorem 6.

Definition 4. *Let $\{\beta_{\mathrm{k}}, \beta^{\mathrm{U}}_{\mathrm{k}}\}_{\mathrm{k}=0}^{\mathsf{L}}$ be as in Definition 3. The effective index set $\mathrm{J} \cup \mathrm{J}^{\mathrm{E}}$ is defined by*

$$\boxed{\mathrm{J} := \left\{\mathrm{k} \in \mathbb{I}^0_{\mathsf{L}} \mid \beta_{\mathrm{k}} < \beta^{\mathrm{U}}_{\mathrm{k}}\right\} \quad \text{and} \quad \mathrm{J}^{\mathrm{E}} := \left\{\mathrm{m} \in \mathbb{I}^0_{\mathsf{L}} \mid \beta_{\mathrm{m}} = \beta^{\mathrm{U}}_{\mathrm{m}}\right\} \, .} \tag{29}$$

The entries J_{k} of J are ordered as it follows:

$$\mathrm{J} = \{\mathrm{J}_0, \mathrm{J}_1, \ldots, \mathrm{J}_{\mathrm{p}}\} \quad \text{where} \quad \mathrm{p} := \sharp\,\mathrm{J} - 1 \quad \text{and} \quad \mathrm{J}_{\mathrm{k}-1} < \mathrm{J}_{\mathrm{k}} \quad \forall\, \mathrm{k} \, . \tag{30}$$

Using Definition 3 and Remark 6,

$$(\mathrm{J}_0 = 0,\ \mathrm{J}_{\mathrm{p}} = \mathsf{L}) \in \mathrm{J}^2 \quad \text{with} \quad \beta_{\mathrm{J}_{-1}} := \beta^{\mathrm{U}}_{\mathrm{J}_0} \equiv \beta^{\mathrm{U}}_0 = +\infty \quad \text{and} \quad \beta_{\mathrm{J}_{\mathrm{p}}} \equiv \beta_{\mathsf{L}} = 0 \, . \tag{31}$$

It worths emphasizing that the set J is always nonempty (Remark 6). The superscript $^{\mathrm{E}}$ in J^{E} evokes equality.

The proofs of several statements in this subsection are delegated to Appendix B.2. The next claim is a cautionary consequence of Theorems 4 and 6.

Lemma 6. *Let H1 hold. One has* $\quad \widehat{\mathrm{R}} \cap \widehat{\mathrm{C}}_{\mathrm{k}} = \varnothing \quad$ *if and only if* $\quad \mathrm{k} \in \mathbb{I}_{\mathsf{L}}^{0} \setminus \{\mathrm{J} \cup \mathrm{J}^{\mathrm{E}}\}$.

In words: for any $\beta > 0$, the optimal set $\widehat{\mathrm{R}}$ of problem $(\mathcal{R}_{\beta})$ does not contain optimal solutions of $(\mathcal{C}_{\mathrm{k}})$, $\mathrm{k} \leqslant \mathsf{L}$, unless k belongs to $\mathrm{J} \cup \mathrm{J}^{\mathrm{E}}$. Thus $\;\widehat{\mathrm{R}} = \bigcup_{\mathrm{k} \in \{\mathrm{J} \cup \mathrm{J}^{\mathrm{E}}\}} \widehat{\mathrm{C}}_{\mathrm{k}}$.

A simplification of the parameters $\{\beta_{\mathrm{k}},\, \beta_{\mathrm{k}}^{\mathrm{U}}\}_{\mathrm{k} \in \mathrm{J} \cup \mathrm{J}^{\mathrm{E}}}$ is derived in Proposition 3.

Proposition 3. *Let H1 hold,* $\{\beta_{\mathrm{k}}, \beta_{\mathrm{k}}^{\mathrm{U}}\}$ *and* J *be as in Definition 3 and Definition 4, respectively. Then*

(a) $\quad \beta_{\mathrm{J}_{\mathrm{k}}} < \beta_{\mathrm{J}_{\mathrm{k}}}^{\mathrm{U}} = \beta_{\mathrm{J}_{\mathrm{k}-1}} \qquad \forall\, \mathrm{J}_{\mathrm{k}} \in \mathrm{J} \setminus \{\mathrm{J}_0\} \quad \text{and} \quad \beta_{\mathrm{J}_0^{\mathrm{U}}} \equiv \beta_{\mathrm{J}_{-1}} = +\infty$.

(b) $\quad \beta_{\mathrm{J}_{\mathrm{k}}} = \dfrac{\mathrm{c}_{\mathrm{J}_{\mathrm{k}}} - \mathrm{c}_{\mathrm{J}_{\mathrm{k}+1}}}{\mathrm{J}_{\mathrm{k}+1} - \mathrm{J}_{\mathrm{k}}} \qquad \forall\, \mathrm{J}_{\mathrm{k}} \in \mathrm{J} \setminus \{\mathrm{J}_{\mathrm{p}}\} \quad \text{and} \quad \beta_{\mathrm{J}_{\mathrm{p}}} \equiv \beta_{\mathsf{L}} = 0$.

(c) $\quad \{\, \beta_{\mathrm{m}} \mid \mathrm{m} \in \mathrm{J}^{\mathrm{E}}\} \subset \big\{\beta_{\mathrm{J}_{\mathrm{k}}} \;\big|\; \mathrm{J}_{\mathrm{k}} \in \mathrm{J} \setminus \{\mathrm{J}_{\mathrm{p}}\}\big\}$.

Proof. (a)-(b) Let $(\mathrm{J}_{\mathrm{k}-1}, \mathrm{J}_{\mathrm{k}}) \in \mathrm{J}^2$. Applying Definition 3 for $\beta_{\mathrm{J}_{\mathrm{k}-1}}$ and for $\beta_{\mathrm{J}_{\mathrm{k}}}^{\mathrm{U}}$ yields

$$\beta_{\mathrm{J}_{\mathrm{k}}}^{\mathrm{U}} \leqslant \frac{\mathrm{c}_{\mathrm{J}_{\mathrm{k}-1}} - \mathrm{c}_{\mathrm{J}_{\mathrm{k}}}}{\mathrm{J}_{\mathrm{k}} - \mathrm{J}_{\mathrm{k}-1}} \leqslant \beta_{\mathrm{J}_{\mathrm{k}-1}} \ . \tag{32}$$

Assume that $\beta_{\mathrm{J}_{\mathrm{k}}}^{\mathrm{U}} < \beta_{\mathrm{J}_{\mathrm{k}-1}}$ and that $(\mathrm{m}_1, \ldots, \mathrm{m}_{\mathrm{q}}) \in (\mathrm{J}^{\mathrm{E}})^{\mathrm{q}}$ satisfy $\beta_{\mathrm{m}_{\mathrm{i}}} \in (\beta_{\mathrm{J}_{\mathrm{k}}}^{\mathrm{U}}, \beta_{\mathrm{J}_{\mathrm{k}-1}})$, $\forall\, \mathrm{i} \in \mathbb{I}_{\mathrm{q}}$. Since $\widehat{\mathrm{R}}_{\beta} \neq \varnothing \;\; \forall\, \beta > 0$, Proposition 1(c) implies that for $\beta \in \Big(\beta_{\mathrm{J}_{\mathrm{k}}}^{\mathrm{U}},\, \beta_{\mathrm{J}_{\mathrm{k}-1}}\Big) \setminus \{\beta_{\mathrm{m}_{\mathrm{i}}}\}_{\mathrm{i}=1}^{\mathrm{q}}$ there is $\mathrm{n} \in \mathbb{I}_{\mathsf{L}}^{0} \setminus \{\mathrm{J} \cup \mathrm{J}^{\mathrm{E}}\}$ obeying $\widehat{\mathrm{C}}_{\mathrm{n}} \subset \widehat{\mathrm{R}}_{\beta}$, in contradiction to Lemma 6. Therefore, $\beta_{\mathrm{J}_{\mathrm{k}}}^{\mathrm{U}} = \beta_{\mathrm{J}_{\mathrm{k}-1}}$. This, together with (32) and the formula of J in (29) gives that

$$\beta_{\mathrm{J}_{\mathrm{k}}} < \beta_{\mathrm{J}_{\mathrm{k}}}^{\mathrm{U}} = \frac{\mathrm{c}_{\mathrm{J}_{\mathrm{k}-1}} - \mathrm{c}_{\mathrm{J}_{\mathrm{k}}}}{\mathrm{J}_{\mathrm{k}} - \mathrm{J}_{\mathrm{k}-1}} = \beta_{\mathrm{J}_{\mathrm{k}-1}} \ .$$

(c) Let $\mathrm{m} \in \mathrm{J}^{\mathrm{E}}$. By Definition 4, $\beta_{\mathrm{m}} > \beta_{\mathrm{J}_{\mathrm{p}}} = 0$. From Theorem 6(b) and statement (a), $\beta_{\mathrm{m}} \notin \big(\beta_{\mathrm{J}_{\mathrm{k}}}, \beta_{\mathrm{J}_{\mathrm{k}-1}}\big)$ for any $\mathrm{J}_{\mathrm{k}} \in \mathrm{J}$. Consequently, there exists $\mathrm{J}_{\mathrm{k}} \in \mathrm{J} \setminus \{\mathrm{J}_{\mathrm{p}}\}$ such that $\beta_{\mathrm{m}} = \beta_{\mathrm{J}_{\mathrm{k}}}$. □

We emphasize that $\{\beta_{\mathrm{k}}\}_{\mathrm{k} \in \mathrm{J}}$ is strictly decreasing and that its first entry is β_0.

In Example 3 we designed a decreasing sequence $\{\mathrm{c}_{\mathrm{k}}\}_{\mathrm{k}=0}^{\mathsf{L}}$ that illustrates several special cases.

Example 3. Let $\{\mathrm{c}_{\mathrm{k}}\}_{\mathrm{k}=0}^{\mathsf{L}}$ for $\mathsf{L} = 7$ reads as

$$\mathrm{c}_0 = 48 \quad \mathrm{c}_1 = 40 \quad \mathrm{c}_2 = 30 \quad \mathrm{c}_3 = 22 \quad \mathrm{c}_4 = 14 \quad \mathrm{c}_5 = 10 \quad \mathrm{c}_6 = 4 \quad \mathrm{c}_7 = 0 \ . \tag{33}$$

According to Definition 3 the sequences $\{\beta_{\mathrm{k}}, \beta_{\mathrm{k}}^{\mathrm{U}}\}_{\mathrm{k}=0}^{7}$ are given by

$$\begin{array}{cccccccc} \boldsymbol{\beta_0 = 9} & \beta_1 = 10 & \boldsymbol{\beta_2 = 8} & \beta_3 = 8 & \boldsymbol{\beta_4 = 5} & \beta_5 = 6 & \boldsymbol{\beta_6 = 4} & \boldsymbol{\beta_7 = 0} \\ \beta_0^{\mathrm{U}} = +\infty & \beta_1^{\mathrm{U}} = 8 & \beta_2^{\mathrm{U}} = \mathbf{9} & \beta_3^{\mathrm{U}} = 8 & \beta_4^{\mathrm{U}} = \mathbf{8} & \beta_5^{\mathrm{U}} = 4 & \beta_6^{\mathrm{U}} = \mathbf{5} & \beta_7^{\mathrm{U}} = \mathbf{4} \end{array} \tag{34}$$

From Definition 4, p = 4, $\quad \mathrm{J} = \{\, \mathrm{J}_0 = \mathbf{0},\, \mathrm{J}_1 = \mathbf{2},\, \mathrm{J}_2 = \mathbf{4},\, \mathrm{J}_3 = \mathbf{6},\, \mathrm{J}_4 = \mathbf{7}\}$ and $\mathrm{J}^{\mathrm{E}} = \{\, \mathbf{3}\,\}$.

– One has $\beta_{\mathrm{J}_{\mathrm{k}}} = \beta_{\mathrm{J}_{\mathrm{k}+1}}^{\mathrm{U}}$ for any $\mathrm{J}_{\mathrm{k}} \in \mathrm{J}$ as asserted in Proposition 3(a).

– The formula in Proposition 3(b) is easy to verify.

– $\{\beta_3 \mid 3 \in \mathrm{J^E}\}$ yields $\beta_3 = \beta_{\mathrm{J}_1} = 8$ and thus $\{\beta_3 \mid 3 \in \mathrm{J^E}\} \subset \{\beta_{\mathrm{J_k}} \mid \mathrm{J_k} \in \mathrm{J} \setminus \{\mathrm{J}_4\}\}$ (Proposition 3(c)).

– One has $\mathrm{J}^{\mathrm{E}}_{\beta_{\mathrm{J}_1}} := \{\mathrm{m} \in \mathrm{J^E} \mid \beta_\mathrm{m} = \beta_{\mathrm{J}_1}\} = \{3 \in \mathrm{J^E} \mid \mathrm{J}_1 < 3 < \mathrm{J}_2\}$, as seen in Lemma 7.

– Observe that J has the smallest indexes so that $\{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathrm{J}} = \{\,9,\ 8,\ 5,\ 4,\ 0\,\}$ is the longest strictly decreasing subsequence of $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{7}$ containing β_0 – see Proposition 4. Another set yielding the same $\{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathrm{J}}$ is $\mathrm{J}' := \{\,0,\ 3,\ 4,\ 6,\ 7\}$; however its indexes are larger than those of J: $\mathrm{J}'_2 > \mathrm{J}_2$. $\diamondsuit$

The location of $\{\beta_\mathrm{m} \,|\, \mathrm{m} \in \mathrm{J^E}\}$ is given by the (probably empty) subsets

$$\mathrm{J}^{\mathrm{E}}_{\beta_{\mathrm{J_k}}} := \{\mathrm{m} \in \mathrm{J^E} \mid \beta_\mathrm{m} = \beta_{\mathrm{J_k}}\}\ . \tag{35}$$

Lemma 7. *Let H1 hold. The sets* $\mathrm{J}^{\mathrm{E}}_{\beta_{\mathrm{J_k}}}$ *in* (35) *fulfill* $\mathrm{J}^{\mathrm{E}}_{\beta_{\mathrm{J_k}}} = \varnothing$ *for* $\mathrm{k} = \mathrm{p}$*, and for any* $\mathrm{k} \leqslant \mathrm{p} - 1$

$$\mathrm{J}^{\mathrm{E}}_{\beta_{\mathrm{J_k}}} = \{\mathrm{m} \in \mathrm{J^E} \mid \mathrm{J_k} < \mathrm{m} < \mathrm{J_{k+1}}\}\ . \tag{36}$$

We want to know how J and $\{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathrm{J}}$ are related to $\{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathbb{I}^0_\mathsf{L}}$ in (25), Definition 3. Both J and $\{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathrm{J}}$ are characterized in the following proposition:

Proposition 4. *Let H1 hold,* $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ *read as in Definition 3 and* J *as in Definition 4. Then* $0 \in \mathrm{J}$ *and* J *contains the smallest indexes such that* $\{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathrm{J}}$ *is the longest strictly decreasing subsequence of* $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ *containing* β_0*.*

In order to find the effective indexes J and values $\{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathrm{J}}$ we need only $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ in (25), Definition 3.

4 Equivalence relations between the optimal sets of $(\mathcal{C}_\mathrm{k})$ and $(\mathcal{R}_\beta)$

In this section we derive the main results of this paper. The most general relationship between the global minimizers of $(\mathcal{C}_\mathrm{k})$ and $(\mathcal{R}_\beta)$ is a *partial equivalence* formulated in Theorems 7 and 8 (subsection 4.1). According to the content of the data d, *quasi-complete equivalence* can hold in the sense that the global minimizers of both problems differ only at $\beta = \beta_\mathrm{k}$ for $\mathrm{k} = 0, \ldots, \mathsf{L} - 1$. The result is presented in Theorem 9 (subsection 4.2).

4.1 Partial equivalence

The main result of this work can be stated in the following theorems. We recall that the set J cannot be empty (Remark 6).

Theorem 7. *Let H1 hold,* $\{\beta_\mathrm{k}\}$ *be as in Definition 3 and* J *as in Definition 4. Then the following hold:*

$$\Big\{\ \widehat{\mathrm{R}}_\beta \ \big|\ \beta \in \big(\beta_{\mathrm{J_k}},\ \beta_{\mathrm{J_{k-1}}}\big)\ \Big\} = \widehat{\mathrm{C}}_{\mathrm{J_k}} \quad \forall\ \mathrm{J_k} \in \mathrm{J}\ , \tag{37}$$

$$\left(\bigcup_{\mathrm{n}=1}^{\mathrm{p}} \big[\beta_{\mathrm{J_k}},\ \beta_{\mathrm{J_{k-1}}}\big]\right) \cup \big[\beta_{\mathrm{J}_0}, \beta_{\mathrm{J}_{-1}}\big) = [\,0, +\infty)\ . \tag{38}$$

Proof. From Proposition 3(a), $\Big(\beta_{\mathrm{J_k}},\ \beta^{\mathrm{U}}_{\mathrm{J_k}}\Big) = \big(\beta_{\mathrm{J_k}},\ \beta_{\mathrm{J_{k-1}}}\big) \neq \varnothing$ for any $\mathrm{J_k} \in \mathrm{J}$. Then (37) is an immediate consequence of Theorem 6(b). Definition 4 and Proposition 3(a) directly lead to (38). $\square$

Discussion on Theorem 7. The theorem states that for any $\beta \notin \{\beta_{\rm k} \mid {\rm k} \in {\rm J}\}$ there is an optimal set $\widehat{\rm C}_{\rm k}$ of problem $(\mathcal{C}_{\rm k})$ that coincides with the optimal set $\widehat{\rm R}_\beta$ of problem $(\mathcal{R}_\beta)$ for a whole range of parameter values β. More precisely, the effective parameter values $\{\beta_{{\rm J}_0}, \cdots, \beta_{{\rm J}_{\rm p-1}}\}$ in Definition 4 partition the positive axis $(0,+\infty)$ into $\sharp\,{\rm J}$ proper intervals. For any $\beta \in (\beta_{{\rm J}_{\rm k}},\ \beta_{{\rm J}_{\rm k-1}})$ the optimal set of problem $(\mathcal{R}_\beta)$ equals the optimal set of problem $(\mathcal{C}_{\rm n})$ for ${\rm n} = {\rm J}_{\rm k}$. The agreement described above can be referred to as partial equivalence because when $\mathbb{I}_{\mathsf{L}}^0 \setminus {\rm J}$ is nonempty, the optimal sets $(\mathcal{C}_{\rm k})$ for ${\rm k} \in \mathbb{I}_{\mathsf{L}}^0 \setminus {\rm J}$ cannot be optimal solutions of $(\mathcal{R}_\beta)$ for any $\beta > 0$.

Remark 7. Since $\{\beta_{\rm k} \mid {\rm k} \in {\rm J}\setminus\{\mathsf{L}\}\}$ is a finite set of isolated values in $(0,\infty)$, the selection of a β belonging to this set can be considered as extremely exceptional ($\beta \neq \beta_{{\rm J}_{\rm k}}$ for any k is a generic property).

In spite of this remark, we will describe the optimal sets of problem $(\mathcal{R}_\beta)$ for $\beta_{\rm k},\ {\rm k} \in {\rm J}$.

Theorem 8. *Let H1 hold. Let $\{\beta_{\rm k}\}$ be as in Definition 3 and* J *as in Definition 4. Then*

$$\widehat{\rm R}_{\beta_{{\rm J}_{\rm k}}} = \widehat{\rm C}_{{\rm J}_{\rm k}} \cup \widehat{\rm C}_{{\rm J}_{\rm k+1}} \cup \left(\bigcup\nolimits_{{\rm m} \in {\rm J}^{\rm E}_{\beta_{{\rm J}_{\rm k}}}} \widehat{\rm C}_{\rm m} \right) \quad \forall\ {\rm J}_{\rm k} \in {\rm J}\setminus\{{\rm J}_{\rm p}\}\ , \tag{39}$$

where ${\rm J}^{\rm E}_{\beta_{{\rm J}_{\rm k}}}$ *obeys* (36) *and* $\widehat{\rm C}_{\rm k} \cap \widehat{\rm C}_{\rm n} = \varnothing$ *for any* $({\rm k},{\rm n}) \in ({\rm J}\cup{\rm J}^{\rm E})^2\ ,\ {\rm k} \neq {\rm n}$.

Proof. For any ${\rm k} \leqslant {\rm p}-1$, Proposition 3(a) and Theorem 6(a) show that $\widehat{\rm C}_{{\rm J}_{\rm k}} \subseteq \widehat{\rm R}_\beta\ \forall\ \beta \in [\beta_{{\rm J}_{\rm k}}, \beta_{{\rm J}_{\rm k-1}}]$ and that $\widehat{\rm C}_{{\rm J}_{\rm k+1}} \subseteq \widehat{\rm R}_\beta\ \forall\ \beta \in [\beta_{{\rm J}_{\rm k+1}}, \beta_{{\rm J}_{\rm k}}]$. Therefore, $\widehat{\rm C}_{{\rm J}_{\rm k}} \cup \widehat{\rm C}_{{\rm J}_{\rm k+1}} \subseteq \widehat{\rm R}_{\beta_{{\rm J}_{\rm k}}}$. In addition, $\beta_{\rm m} = \beta_{{\rm J}_{\rm k}}$ for any ${\rm m} \in {\rm J}^{\rm E}_{\beta_{{\rm J}_{\rm k}}}$ (Proposition 3(c) and (35)) which yields $\bigcup_{{\rm m}\in{\rm J}^{\rm E}_{\beta_{{\rm J}_{\rm k}}}} \widehat{\rm C}_{\rm m} \subseteq \widehat{\rm R}_{\beta_{{\rm J}_{\rm k}}}$. The conditions in Theorem 6 for $\beta = \beta_{{\rm J}_{\rm k}}$ can be satisfied only for ${\rm n} \in \left\{{\rm J}_{\rm k} \cup {\rm J}^{\rm E}_{\beta_{{\rm J}_{\rm k}}}\right\}$ because $\{\beta_{{\rm J}_{\rm k}}\}$ is strictly decreasing (Proposition 3(a)). Hence the equality in (39). The set ${\rm J}^{\rm E}_{\beta_{{\rm J}_{\rm k}}}$ satisfies (36) by Lemma 7. The result on the intersection of the sets $\widehat{\rm C}_{\rm n}$ comes from Corollary 1. □

Example 4. [Example 3, continued] Let $\{\beta_{\rm k}\}_{\rm k=0}^7$, J and ${\rm J}^{\rm E}$ be as in Example 3. We recall that ${\rm J} = \{0,2,4,6,7\}$ and that ${\rm J}^{\rm E} = \{\,3\,\}$, so ${\rm J}^{\rm E}_2 = \{3\}$ and ${\rm J}^{\rm E}_{\rm k} = \varnothing$ otherwise. By Theorems 7 and 8 one has

$$\{\,\widehat{\rm R}_\beta \mid \beta > 9\} = \widehat{\rm C}_0 \quad \{\,\widehat{\rm R}_\beta \mid \beta \in (8,9)\} = \widehat{\rm C}_2 \quad \{\,\widehat{\rm R}_\beta \mid \beta \in (5,8)\} = \widehat{\rm C}_4 \quad \{\,\widehat{\rm R}_\beta \mid \beta \in (4,5)\} = \widehat{\rm C}_6 \quad \{\,\widehat{\rm R}_\beta \mid \beta \in (0,4)\} = \widehat{\rm C}_7$$

$$\text{and} \quad \widehat{\rm R}_{\beta=9} = \widehat{\rm C}_0 \cup \widehat{\rm C}_2 \quad \widehat{\rm R}_{\beta=8} = \widehat{\rm C}_2 \cup \widehat{\rm C}_3 \cup \widehat{\rm C}_4 \quad \widehat{\rm R}_{\beta=5} = \widehat{\rm C}_4 \cup \widehat{\rm C}_6 \quad \widehat{\rm R}_{\beta=4} = \widehat{\rm C}_6 \cup \widehat{\rm C}_7\ .$$

Discussion on Theorem 8. Proposition 2 have proved that the sets ${\rm J}^{\rm E}_{\beta_{{\rm J}_{\rm k}}}$ are empty with an overwhelming probability. So (39) in Theorem 7 shows that for $\beta \in \{\beta_{\rm k} \mid {\rm k} \in {\rm J}\setminus\{\mathsf{L}\}\}$, the optimal set of problem $(\mathcal{R}_\beta)$ is normally composed out of two optimal sets of problem $(\mathcal{C}_{\rm k})$, namely $\widehat{\rm R}_{\beta_{{\rm J}_{\rm k}}} = \widehat{\rm C}_{{\rm J}_{\rm k}} \cup \widehat{\rm C}_{{\rm J}_{\rm k+1}}$. Further, if an ${\rm J}^{\rm E}_{\beta_{{\rm J}_{\rm k}}}$ is nonempty, we know from Lemma 7 that the optimal set of $(\mathcal{R}_\beta)$ for $\beta = \beta_{{\rm J}_{\rm k}}$ can involve at most ${\rm J}_{\rm k+1} - {\rm J}_{\rm k} - 1$ additional optimal sets $\widehat{\rm C}_{\rm m}$ where ${\rm m} \in ({\rm J}_{\rm k}, {\rm J}_{\rm k+1})$.

A partial equivalence between problems $(\mathcal{C}_{\rm k})$ and $(\mathcal{R}_\beta)$ always exists.

For the $\sharp\,{\rm J}-1$ isolated values $\{\beta_{\rm k} \mid {\rm k} \in {\rm J}\setminus\{\mathsf{L}\}\}$ problem $(\mathcal{R}_{\beta_{\rm k}})$ has normally two optimal sets.

4.2 Quasi-complete equivalence

Here we explore the conditions enabling the equivalence results in Theorem 7 to hold for any $\mathrm{k} \leqslant \mathsf{L}$.

Lemma 8. *Let H1 hold. Let* J *be as in Definition 4. Then the following hold:*

(a) *If the sequence* $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ *in Definition 3 is strictly decreasing, then its entries read as*

$$\boxed{\beta_k = \mathrm{c}_\mathrm{k} - \mathrm{c}_{\mathrm{k}+1} \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}-1}^{0} \quad \text{and} \quad \beta_{\mathsf{L}} = 0, \quad \beta_{-1} := \beta_0^{\mathrm{U}} = +\infty\ .} \tag{40}$$

(b) *If the sequence* $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ *in* (40) *is strictly decreasing then* $\mathrm{J} = \mathbb{I}_{\mathsf{L}}^{0}$.

Proof. (a) Since $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ is strictly decreasing, Proposition 4 implies that the set J in (29) obeys $\mathrm{J} = \mathbb{I}_{\mathsf{L}}^{0}$. Applying Proposition 3(b) with $\mathrm{J}_\mathrm{k} = \mathrm{k}$ and $\mathrm{J}_{\mathrm{k}+1} = \mathrm{k}+1$ delivers the formula in (40).

(b) Let $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ in (40) be strictly decreasing. Then β_k in (40) satisfies Proposition 3(b) for any $\mathrm{k} \in \mathrm{J} = \mathbb{I}_{\mathsf{L}}^{0}$. By setting $\beta_\mathrm{k}^{\mathrm{U}} := \beta_{\mathrm{k}-1}$, one has $\mathrm{J} = \mathbb{I}_{\mathsf{L}}^{0}$ according to Definition 4. □

Theorem 9. *Let H1 hold. Let* $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ *in* (40) *be strictly decreasing. Then*

$$\Big\{ \ \widehat{\mathrm{R}}_\beta \ \Big| \ \ \beta \in (\,\beta_\mathrm{k}, \beta_{\mathrm{k}-1}) \Big\} = \widehat{\mathrm{C}}_\mathrm{k} \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}}^{0} \tag{41}$$

$$\widehat{\mathrm{R}}_{\beta_\mathrm{k}} = \widehat{\mathrm{C}}_\mathrm{k} \cup \widehat{\mathrm{C}}_{\mathrm{k}+1} \quad \textit{with} \quad \widehat{\mathrm{C}}_\mathrm{k} \cap \widehat{\mathrm{C}}_{\mathrm{k}+1} = \varnothing \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}-1}^{0}\ . \tag{42}$$

Proof. Since $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ in (40) is strictly decreasing, one has $\mathrm{J} = \mathbb{I}_{\mathsf{L}}^{0}$ by Lemma 8(b). So, $\mathrm{J}^{\mathrm{E}} = \varnothing$ in Definition 4 and $\mathrm{J}_\mathrm{k}^{\mathrm{E}} = \varnothing$ in (35). Then the statement follows from Theorems 7 and 8. □

Discussion on Theorem 9. The condition that $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$ in (40) is strictly decreasing reads as

$$\beta_{\mathrm{k}-1} > \beta_\mathrm{k} \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}-1} \quad \Longleftrightarrow \quad \mathrm{c}_{\mathrm{k}-1} - \mathrm{c}_\mathrm{k} > \mathrm{c}_\mathrm{k} - \mathrm{c}_{\mathrm{k}+1} \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}-1}\ . \tag{43}$$

For generic data, $\{\mathrm{c}_\mathrm{k}\}$ is strictly decreasing (Proposition 5(b)). For such generic data and $\mathrm{k} \leqslant \mathsf{L}-1$, let $\overline{u}$, $\widehat{u}$ and $\widetilde{u}$ be optimal solutions of problems $(\mathcal{C}_{\mathrm{k}-1})$, $(\mathcal{C}_\mathrm{k})$ and $(\mathcal{C}_{\mathrm{k}+1})$, respectively. Denote $\overline{\sigma} := \mathrm{supp}(\overline{u})$, $\widehat{u} := \mathrm{supp}(\widehat{u})$ and $\widetilde{\sigma} := \mathrm{supp}(\widetilde{u})$. The condition on the right hand side in (43) reads

$$d^{\mathrm{T}}\,(\Pi_{\widehat{\sigma}} - \Pi_{\overline{\sigma}})\,d > d^{\mathrm{T}}\,(\Pi_{\widetilde{\sigma}} - \Pi_{\widehat{\sigma}})\,d \tag{44}$$

where both sides in the inequality above are positive. It is reasonable to expect that there are data d such that the above condition holds for any $\mathrm{k} \leqslant \mathsf{L}$. Note such data would belong to an open subset of $\mathbb{R}^{\mathsf{M}}$ where (44) is satisfied. Given a series of numerical tests on 5×10 matrices (see section 7) the realization of (43) essentially depends on d.

Theorem 9 shows that it is *possible* to have equivalence between problems $(\mathcal{C}_\mathrm{k})$ and $(\mathcal{R}_\beta)$, except for the isolated parameter values $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{L}}$. If the matrix A is specified and if there are assumptions on the data, one could infer knowledge whether the context of Theorem 9 is a regular regime or not. If the answer turns to be positive, this can be used in the development of numerical schemes.

A mid-way scenario appears as an immediate consequence of Proposition 3(b), (35) and Theorem 7.

Remark 8. Let H1 hold, $\{\beta_{\mathrm{k}}, \beta_{\mathrm{k}}^{\mathrm{U}}\}$ and J be as in Definition 3 and Definition 4, respectively. Suppose that for $\mathrm{m} \geqslant 0$ and $\mathrm{n} \geqslant 1$ the entries of J satisfy

$$\mathrm{J}' := \{\mathrm{J_m}, \dots, \mathrm{J_{m+n}}\} = \{\mathrm{J_m}, \mathrm{J_m}+1, \mathrm{J_m}+2, \dots, \mathrm{J_m}+\mathrm{n}\} \ .$$

Then

$$\beta_{\mathrm{J_{m+k}}} = \mathrm{c_{J_m+k}} - \mathrm{c_{J_m+k+1}} \quad \Big\{ \widehat{\mathrm{R}}_\beta \mid \beta \in \big(\beta_{\mathrm{J_{m+k}}},\ \beta_{\mathrm{J_{m+k-1}}}\big) \Big\} = \widehat{\mathrm{C}}_{\mathrm{J_m+k}} \quad \text{and} \quad \mathrm{J}^{\mathrm{E}}_{\beta_{\mathrm{J_k}}} = \varnothing \qquad 0 \leqslant \mathrm{k} \leqslant \mathrm{n}-1 \ .$$

If the subset J' contains all sparsity levels of interest in a given situation, we are again in the setup of Theorem 9 and the comments given after it.

5 On the optimal values of ($\mathcal{C}_{\mathrm{k}}$) and ($\mathcal{R}_\beta$)

The proofs of the statements in this section are outlined in Appendix C.

Using the definition of Ω_{k} in (17), we introduce the subsets of $\mathbb{R}^{\mathsf{M}}$ given below:

$$\mathrm{E_k} := \bigcup_{\omega \in \Omega_{\mathrm{k}}} \mathrm{range}\,(A_\omega)^{\perp} \quad \text{and} \quad \mathrm{G_k} := \bigcup_{\omega \in \Omega_{\mathrm{k}}} \mathrm{range}\,(A_\omega) \ . \tag{45}$$

Clearly, $\mathrm{E}_0 = \mathrm{G}_{\mathsf{M}} = \mathbb{R}^{\mathsf{M}}$ and $\mathrm{E}_{\mathsf{M}} = \mathrm{G}_0 = \{0\}$ by H1.

The next Proposition 5 gives results on $\{\mathrm{c_k}\}_{\mathrm{k}=0}^{\mathsf{M}}$ in connection with $d \in \mathbb{R}^{\mathsf{M}}$.

Proposition 5. *Let H1 hold. Let* $\mathsf{L}' \leqslant \mathsf{M}$ *be arbitrarily fixed. Then*

(a) $\quad \mathrm{c_k} > 0 \quad \forall\, \mathrm{k} \leqslant \mathsf{L}'-1 \quad \Longleftrightarrow \quad d \in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{G}_{\mathsf{L}'-1}$;

(b) $\quad d \in \mathbb{R}^{\mathsf{M}} \setminus (\mathrm{E}_2 \cup \mathrm{G}_{\mathsf{L}'-1}) \quad \Longrightarrow \quad \mathrm{c_{k-1}} > \mathrm{c_k} \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}'}$.

Proposition 5(a) shows that the constant L in (15) corresponds to $d \in \mathrm{G}_{\mathsf{L}} \setminus \mathrm{G}_{\mathsf{L}-1}$.

Remark 9. The subsets E_2 and $\mathrm{G}_{\mathsf{M}-1}$ are finite unions of vector subspaces of dimensions $\mathsf{M}-2$ and $\mathsf{M}-1$, respectively. Hence, $d \in \mathbb{R}^{\mathsf{M}} \setminus (\mathrm{E}_2 \cup \mathrm{G}_{\mathsf{M}-1})$ is a generic property (Definition 1).

Therefore, $\{\mathrm{c_k}\}_{\mathrm{k}=0}^{\mathsf{M}}$ is strictly decreasing and $\mathsf{L} = \mathsf{M}$ generically. $\diamondsuit$

By Proposition 1(a) and Theorem 4, for any $\beta > 0$ the optimal value function of problem ($\mathcal{R}_\beta$) in (6), $\beta \mapsto r_\beta = \inf\big\{\mathcal{F}_\beta(u) \mid u \in \mathbb{R}^{\mathsf{N}}\big\}$, equivalently reads as

$$r_\beta = \min\big\{\ \mathrm{c_k} + \beta\,\mathrm{k} \mid \mathrm{k} \in \mathbb{I}^0_{\mathsf{L}}\ \big\} \ . \tag{46}$$

From this observation and Theorems 7 and 8 one infers the following:

Corollary 2. *Let H1 hold and* J *be as in Definition 4. The application* $\beta \mapsto r_\beta\ : (0, +\infty) \to \mathbb{R}$ *fulfills*

(a) $\left\{ \begin{array}{rcl} r_\beta & = & \mathrm{c_{J_k}} + \beta\,\mathrm{J_k} \\ & = & \mathcal{F}_\beta(\widehat{u}) \quad \forall\, \widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{J_k}} \end{array} \right.$ *if and only if* $\beta \in \left\{ \begin{array}{ll} [\beta_{\mathrm{J_0}},\ +\infty) & \text{for} \quad \mathrm{J_0} = 0\ ; \\ \big[\beta_{\mathrm{J_k}},\ \beta_{\mathrm{J_{k-1}}}\big] & \text{for} \quad \mathrm{J_k} \in \mathrm{J} \setminus \{0, \mathsf{L}\}\ ; \\ \big(0,\ \beta_{\mathrm{J_{p-1}}}\big] & \text{for} \quad \mathrm{J_p} = \mathsf{L}\ . \end{array} \right.$

(b) $\beta \mapsto r_\beta$ *is continuous and concave.*

(c) $r_{\beta_{\mathrm{J_{k-1}}}} > r_{\beta_{\mathrm{J_k}}} \quad \forall\, \mathrm{J_k} \in \mathrm{J}\,, \qquad r_{\beta_{\mathrm{J_0}}} = \mathrm{c_{J_0}} = r_\beta \quad \forall\, \beta \geqslant \beta_{\mathrm{J_0}}$ *and* $\quad r_{\beta_{\mathrm{J_0}}} > r_\beta \quad \forall\, \beta < \beta_{\mathrm{J_0}}$.

$\beta \mapsto r_\beta$ is affine increasing on each interval $\left(\beta_{\mathrm{J_k}}, \beta_{\mathrm{J_{k-1}}}\right)$ with upward kinks at $\beta_{\mathrm{J_k}}$ for any $\mathrm{J_k} \in \mathrm{J} \setminus \{\mathsf{L}\}$ and bounded by c_0.

Example 5. [Continuation of Example 3] Let us consider again $\{\mathrm{c_k}\}_{\mathrm{k}=0}^{7}$ in (33) along with $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{7}$ and J in (34). From Corollary 2, the mapping $\beta \mapsto r_\beta$ is given by

$$\begin{array}{lll}
\beta \in (0,4] & r_\beta = \mathrm{c}_7 + 7\beta = 7\beta & r_{\beta=4} = 28 \\
\beta \in [4,5] & r_\beta = \mathrm{c}_6 + 6\beta = 4 + 6\beta & r_{\beta=5} = 34 \\
\beta \in [5,8] & r_\beta = \mathrm{c}_4 + 4\beta = 14 + 4\beta & r_{\beta=8} = 46 \\
\beta \in [8,9] & r_\beta = \mathrm{c}_2 + 2\beta = 30 + 2\beta & r_{\beta=9} = 48 \\
\beta \in [9,+\infty) & r_\beta = \mathrm{c}_0 + 0\beta = 48 &
\end{array}$$

6 Cardinality of the optimal sets of $(\mathcal{R}_\beta)$ and of $(\mathcal{C}_\mathrm{k})$

For the convex surrogates of problems $(\mathcal{C}_\mathrm{k})$ and $(\mathcal{R}_\beta)$ where $\|u\|_0$ is replaced by $\|u\|_1$, it is well known that the optimal sets are convex closed with possibly a continuum of solutions.

Proposition 6. *Let H1 hold. For any $\beta > 0$ and for any $\mathrm{k} \leqslant \mathsf{L}$, the optimal sets $\widehat{\mathrm{R}}_\beta$ and $\widehat{\mathrm{C}}_\mathrm{k}$ are finite.*

Proof. From Remark 4, each $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}$ is the unique minimizer of problem $(\mathcal{P}_\omega)$ for $\omega := \mathrm{supp}(\widehat{u})$ where $\omega \in \Omega_\mathrm{k}$ (Theorem 1(a) and the notation in (17)). Therefore, $\sharp\, \widehat{\mathrm{C}}_\mathrm{k}$ is finite (with $\sharp\, \widehat{\mathrm{C}}_\mathrm{k} \leqslant \sharp\, \Omega_\mathrm{k}$). This, together with Theorem 7 and Theorem 8, shows that $\sharp\, \widehat{\mathrm{R}}_\beta$ is finite for every $\beta > 0$. □

For any $\beta > 0$ and $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0$ the optimal sets of problems $(\mathcal{C}_\mathrm{k})$ and $(\mathcal{R}_\beta)$ are composed out of a certain finite number of isolated (hence strict) minimizers.

6.1 Uniqueness of the global minimizers of $(\mathcal{C}_\mathrm{k})$ and $(\mathcal{R}_\beta)$

If $\mathsf{L} = \mathsf{M}$, Remark 4 shows that $\sharp\, \widehat{\mathrm{C}}_{\mathsf{M}} = \sharp\, \Omega_{\mathsf{M}}$.

Let $\mathrm{k} \leqslant \min\{\mathsf{L}, \mathsf{M}-1\}$ and $(\widehat{u}, \widetilde{u}) \in \left(\widehat{\mathrm{C}}_\mathrm{k}\right)^2$ for $\widehat{u} \neq \widetilde{u}$. Set $\widehat{\sigma} := \mathrm{supp}(\widehat{u})$ and $\widetilde{\sigma} := \mathrm{supp}(\widetilde{u})$. By Theorem 1(a), $(\widehat{\sigma}, \widetilde{\sigma}) \in (\Omega_\mathrm{k})^2$. Then

$$\mathrm{c_k} = \|A_{\widehat{\sigma}} \widehat{u}_{\widehat{\sigma}} - d\|^2 = \|A_{\widetilde{\sigma}} \widetilde{u}_{\widetilde{\sigma}} - d\|^2 \quad \text{where} \quad \widehat{\sigma} \neq \widetilde{\sigma} \; .$$

The expression for $\mathrm{c_k}$ in (20) shows that

$$\|A_{\widehat{\sigma}} \widehat{u}_{\widehat{\sigma}} - d\|^2 - \|A_{\widetilde{\sigma}} \widetilde{u}_{\widetilde{\sigma}} - d\|^2 = d^{\mathrm{T}} \left(\Pi_{\widetilde{\sigma}} - \Pi_{\widehat{\sigma}}\right) d = 0 \; . \tag{47}$$

The last equality in (47) suggests that $\widehat{\mathrm{C}}_\mathrm{k}$ *could* be a singleton under the assumption $\mathrm{H}^\star$ below.

H*. For $\mathsf{K} \leqslant \min\{\mathsf{M}-1, \mathsf{L}\}$ fixed, $A \in \mathbb{R}^{\mathsf{M}\times\mathsf{N}}$ obeys $\quad \Pi_\omega \neq \Pi_{\varpi} \quad \forall\, (\omega, \varpi) \in (\Omega_\mathrm{k})^2\,, \;\; \omega \neq \varpi \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{K}}$.

$\mathrm{H}^\star$ is a generic property of all matrices in $\mathbb{R}^{\mathsf{M}\times\mathsf{N}}$ [30, Theorem 5.3]. Under $\mathrm{H}^\star$, the set Δ_{K} below

$$\Delta_{\mathsf{K}} := \textstyle\bigcup_{\mathrm{k}=1}^{\mathsf{K}} \bigcup_{(\omega,\varpi)\in(\Omega_\mathrm{k})^2} \left\{ g \in \mathbb{R}^{\mathsf{M}} \;\middle|\; \omega \neq \varpi \;\text{ and }\; g \in \ker\left(\Pi_{\varpi} - \Pi_\omega\right) \right\}$$

is a finite union of vector subspaces of dimension $\leqslant \mathsf{M}-1$, so *data generically live in* $\mathbb{R}^{\mathsf{M}} \setminus \Delta_{\mathsf{K}}$.

Remark 10. Let the *two generic assumptions*, A satisfies $\mathrm{H}^\star$ and $d \in \mathbb{R}^{\mathsf{M}} \setminus \Delta_{\mathsf{K}}$, hold. From (47), problem $(\mathcal{C}_\mathrm{k})$ $\forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{K}}$ has a unique optimal solution. Using that $\{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathrm{J}}$ is strictly decreasing, we set $\mathsf{K}' := \max\{\mathrm{k} \in \mathrm{J} \mid \mathrm{k} \leqslant \mathsf{K}\}$. Then Theorems 7 and 8 show that problem $(\mathcal{R}_\beta)$ has a unique optimal solution for any $\beta \in (\beta_{\mathsf{K}'}, +\infty) \setminus \{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathrm{J}}$ and hence generically for any $\beta > \beta_{\mathsf{K}'}$ by Remark 7. ◇

For any $\mathrm{k} \leqslant \min\{\mathsf{M}-1, \mathsf{L}\}$ and for any $\beta > \beta_{\mathrm{k_{p-1}}}$ problems $(\mathcal{C}_\mathrm{k})$ and $(\mathcal{R}_\beta)$ generically have a unique global minimizer.

For the sake of generality, we did not consider the assumptions evoked in this subsection.

7 Numerical tests

Here we present two kind of experiments using matrices $A \in \mathbb{R}^{\mathsf{M}\times\mathsf{N}}$ for $(\mathsf{M},\mathsf{N}) = (5, 10)$, original vectors $u^\mathrm{o} \in \mathbb{R}^\mathsf{N}$ and data samples $d = Au^\mathrm{o}(+\text{noise})$ with *two different goals*:

- to get a rough idea on behaviour of the parameters β_k in Definitions 3 and 4;
- to verify and illustrate our theoretical findings.

All results were calculated using an exhaustive combinatorial search.

7.1 Monte Carlo experiments for $(\mathsf{M},\mathsf{N}) = (5, 10)$

We realized two experiments, each one composed of $\mathbf{10^5}$ trials with $(\mathsf{M},\mathsf{N}) = (5,\ 10)$. In each trial, the "original" $u^\mathrm{o} \in \mathbb{R}^\mathsf{N}$ had a random support on $\{1, \ldots, \mathsf{N}\}$ satisfying $\|u^\mathrm{o}\|_0 \leqslant \mathsf{M} - 1 = 4$ with mean 3.79. The coefficients of each A and the non-zero entries of each u^o were independent and identically distributed (i.i.d.). Data were obtained as $d = Au^\mathrm{o}+$ i.i.d. centered Gaussian noise. In each trial we computed the exact optimal values $\{\mathrm{c_k}\}$ and then computed $(\beta_\mathrm{k}, \beta_\mathrm{k}^\mathrm{U})$ according to Definition 3. We considered two different distributions for A and for the non-zero entries of u^o.

– **Experiment $\mathcal{N}(\mathbf{0}, \mathbf{10})$.** All coefficients of each A and all non-zero entries of u^o had a normal distribution with mean 0 and variance 10. The SNR in dB was in [10, 61] with mean value 33.75 dB.
– **Experiment Uni $[\mathbf{0}, \mathbf{10}]$.** The coefficients of A and of $u^\mathrm{o}_{\mathrm{supp}(u^\mathrm{o})}$ were uniform on $[0, 10]$. We had SNR in [20, 55] with a mean of 28.95 dB.

In these experiments, the following facts were observed:

- We had $\mathsf{L} = \mathsf{M}$ in each trial which confirms Proposition 5(b) and Remark 9;
- $\{\mathrm{c_k}\}_{\mathrm{k}=0}^{M}$ was always strictly decreasing – as expected from Proposition 5;
- We never found $\beta_\mathrm{m} = \beta_\mathrm{m}^\mathrm{U}$, so the set $\mathrm{J^E}$ in (29) was always empty; see Proposition 2.
- For every A there were data d so that the sequence $\{\mathrm{c_k} - \mathrm{c_{k-1}}\}_{\mathrm{k}=0}^{\mathsf{L}}$, see (40), was strictly decreasing.

The other results in percentage are shown in Table 1 where N_k reads as

$$N_\mathrm{k} := \sharp \left\{\mathrm{k} \in \mathbb{I}_\mathsf{M}^0 \mid \beta_\mathrm{k} > \beta_{\mathrm{k}-1}\right\} \ . \tag{48}$$

In both experiments, the sequence $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{M}}$ in Definition 3 was strictly decreasing in a huge amount of cases; by Lemma 8(a) in all these cases $\{\beta_\mathrm{k}\}_{\mathrm{k}=0}^{\mathsf{M}}$ equals the sequence in (40) and the quasi-complete equivalence in Theorem 9 holds. One should suppose that these percentages are high because of the small size of the matrices. Anyway, these percentages clearly depend on the distribution of the coefficients of (A, d).

Table 1: Results on the behaviour of $\{\beta_{\rm k}\}$ in Definition 3 for two experiments, each one composed of 10^5 random trials. For $\rm k \geqslant 3$ we had $N_{\rm k} = 0$.

	$\beta_{\rm k} < \beta_{\rm k-1}, \forall\, {\rm k} \in \mathbb{I}^0_{\mathsf{M}}$	$N_{\rm k} = 1$	$N_{\rm k} = 2$	mean(SNR)
$\mathcal{N}(0, 10)$	**93.681 %**	6.254 %	0.065 %	33.75
Uni $[0, 10]$	**98.783 %**	1.216 %	0.001 %	28.95

7.2 Tests on (partial) equivalence with a selected matrix and selected data

Next we present in detail three experiments for $(\mathsf{M}, \mathsf{N}) = (5, 10)$ where

$$
\begin{aligned}
A &= \begin{pmatrix}
13.94 & 16.36 & 4.88 & -3.09 & -15.42 & 1.31 & -3.18 & -12.13 & -4.26 & -10.09 \\
7.06 & -6.48 & -9.07 & -8.37 & -2.72 & -17.42 & -5.83 & -3.81 & 3.87 & -1.80 \\
11.63 & 6.73 & -4.75 & -6.28 & 3.42 & 6.68 & -1.64 & 13.23 & 9.03 & -20.27 \\
-7.54 & 12.74 & -6.66 & 5.01 & 4.84 & 8.98 & -9.35 & 3.85 & 7.18 & 4.09 \\
3.22 & -10.40 & -5.02 & 16.70 & 9.53 & -5.49 & 11.88 & -3.62 & 17.36 & 7.34
\end{pmatrix} \\
u^{\rm o} &= \begin{pmatrix} 0 & \mathbf{4} & 0 & 0 & 0 & \mathbf{9} & 0 & 0 & \mathbf{3} & 0 \end{pmatrix}^{\rm T} .
\end{aligned} \tag{49}
$$

The entries of A follow a nearly normal distribution. The coefficients of A, $u^{\rm o}$, and d in (50), (52) and (54) are *exact*. H1 holds since ${\rm rank}(A) = \mathsf{M} = 5$. Problem $(\mathcal{C}_{\mathsf{M}})$ has $\sharp\, \Omega_{\mathsf{M}} = 252$ optimal solutions; none of them is shown. We have $\beta_0 < \beta_0^{\rm U} = +\infty$ (Remark 6), so $\widehat{\rm C}_0 = \left\{ \widehat{\rm R}_\beta \,|\, \beta > \beta_0 \right\}$ in all cases (Theorem 6). In the tests presented below the optimal set of $(\mathcal{C}_{\rm k})$ for $\rm k \leqslant \mathsf{M} - 1$ is a singleton (see subsection 6.1).

In order to illustrate various cases of partial or quasi-complete equivalence, we selected a couple $(A, u^{\rm o})$ in (49) that behaves differently compared to Table 1: it does not favor quasi-complete equivalence as seen from the $\mathbf{10^5}$ random trials summarized in Table 2.

Table 2: The behaviour of $\{\beta_{\rm k}\}$ in Definition 3 for an experiment with 10^5 trials where A and $u^{\rm o}$ are given by (49) $d = Au^{\rm o}+$ i.i.d. centered Gaussian noise. We had $N_{\rm k} = 0$, $\forall\, \rm k \geqslant 3$.

	$\beta_{\rm k} < \beta_{\rm k-1}, \forall\, {\rm k} \in \mathbb{I}^0_{\mathsf{M}}$	$N_{\rm k} = 1$	$N_{\rm k} = 2$	mean(SNR)
$\boldsymbol{A}$**,** $\boldsymbol{u^{\rm o}}$ **in** (49)	**29.41 %**	70.59 %	0 %	36.25

Noise-free data According to (49), data read as

$$
d = Au^{\rm o} = \begin{pmatrix} 64.45 & -171.09 & 114.13 & 153.32 & -38.93 \end{pmatrix}^{\rm T} . \tag{50}
$$

Since data are noise-free and $\|u^{\rm o}\|_0 = 3$, clearly $\widehat{u} = u^{\rm o}$ is an optimal solution to problems $(\mathcal{C}_{\rm k})$ with ${\rm c_k} = 0$ for ${\rm k} \in \{3, 4, 5\}$ and $\mathsf{L} = 3$. The other optimal values ${\rm c_k}$ are seen in Table 3. By Theorem 4, any $\widehat{u} \in \widehat{\rm R}$ obeys $\|\widehat{u}\|_0 \leqslant 3$. The critical parameters $\{\beta_{\rm k}\}$ by Definition 3 are

$$
\boldsymbol{\beta_3} = 0 < \beta_3^{\rm U} = \boldsymbol{\beta_1} = 3872.46 < \beta_1^{\rm U} = \boldsymbol{\beta_0} = 63729 \quad \text{and} \quad \beta_2 = 3968 > \beta_2^{\rm U} = 3776.82 \, . \tag{51}
$$

$\beta_{\rm k} > \beta_{\rm k}^{\rm U}$ only for $\rm k = 2$, so ${\rm J} = \{0, 1, 3\}$ in (29). By Lemma 6, $\widehat{\rm R} \cap \widehat{\rm C}_{\rm k} = \varnothing$ for $\rm k = 2$. By Theorem 7, $\widehat{\rm C}_3 = \left\{ \widehat{\rm R}_\beta \mid \beta \in (\beta_3, \beta_1) \right\}$ and $\widehat{\rm C}_1 = \left\{ \widehat{\rm R}_\beta \mid \beta \in (\beta_1, \beta_0) \right\}$. The numerical results are seen in Table 3.

Table 3: The optimal values $\mathrm{c_k}$ and the optimal sets of $(\mathcal{C}_\mathrm{k})$ for $\mathrm{k} \in \mathbb{I}_3^0$ where d is as in (50). The values of β_k are given in (51). We recall that $\widehat{\mathrm{R}}_\beta$ is the optimal set of problem $(\mathcal{R}_\beta)$.

k	$\mathrm{c_k}$	$\widehat{\mathrm{C}}_\mathrm{k}$ = the optimal solution of $(\mathcal{C}_\mathrm{k})$, singleton										$\widehat{\mathrm{C}}_\mathrm{k} = \widehat{\mathrm{R}}_\beta$
3	0	0	**4**	0	0	0	**9**	0	0	**3**	0	$\beta \in (\beta_3, \beta_1)$
2	3968	0	**3.25**	0	0	0	**9.29**	0	0	0	0	**no**
1	7745	0	0	0	0	0	**11.76**	0	0	0	0	$\beta \in (\beta_1, \beta_0)$
0	71474	0	0	0	0	0	0	0	0	0	0	$\beta > \beta_0$

Noisy data 1. Data are corrupted with nearly normal, centered, i.i.d. noise and SNR= 32.32 dB:

$$d = \begin{pmatrix} 69.13 & -171.95 & 113.74 & 150.27 & -36.09 \end{pmatrix}^\mathrm{T} . \tag{52}$$

The optimal values $\mathrm{c_k}$ of problems $(\mathcal{C}_\mathrm{k})$ in Table 4 with $\mathrm{c}_5 = 0$ yield $\mathsf{L} = \mathsf{M} = 5$. From Definition 3,

$$\boldsymbol{\beta_5} = 0 < \beta_5^\mathrm{U} = \boldsymbol{\beta_4} = 0.068 < \beta_4^\mathrm{U} = \boldsymbol{\beta_3} = 36.25 < \beta_3^\mathrm{U} = \boldsymbol{\beta_1} = 3987.68 < \beta_1^\mathrm{U} = \boldsymbol{\beta_0} = 63154 \ , \tag{53}$$

while $\beta_2 = 4002.83 > \beta_2^\mathrm{U} = 3972.54$. Hence, $\mathrm{J} = \mathbb{I}_5^0 \setminus \{2\}$ in (29) and $\{\beta_\mathrm{k}\}_{\mathrm{k}\in\mathrm{J}}$ confirms Propositions 3 and 4. By Lemma 6, $\widehat{\mathrm{R}} \cap \widehat{\mathrm{C}}_2 = \varnothing$ and by Theorem 7, $\widehat{\mathrm{C}}_5 = \left\{ \widehat{\mathrm{R}}_\beta \,|\, \beta \in (0, \beta_4) \right\}$, $\widehat{\mathrm{C}}_4 = \left\{ \widehat{\mathrm{R}}_\beta \,|\, \beta \in (\beta_4, \beta_3) \right\}$, $\widehat{\mathrm{C}}_3 = \left\{ \widehat{\mathrm{R}}_\beta \,|\, \beta \in (\beta_3, \beta_1) \right\}$ and $\widehat{\mathrm{C}}_1 = \left\{ \widehat{\mathrm{R}}_\beta \,|\, \beta \in (\beta_1, \beta_0) \right\}$. The numerical tests are shown in Table 4.

Table 4: The optimal values $\mathrm{c_k}$ and the optimal solutions of $(\mathcal{C}_\mathrm{k})$ for $\mathrm{k} \in \mathbb{I}_4^0$ where d is given in (52). The values of β_k are given in (53). We recall that $\widehat{\mathrm{R}}_\beta$ is the set of the global minimizers of $\mathcal{F}_\beta$.

k	$\mathrm{c_k}$	$\widehat{\mathrm{C}}_\mathrm{k}$ = the optimal solution of $(\mathcal{C}_\mathrm{k})$, singleton										$\widehat{\mathrm{C}}_\mathrm{k} = \widehat{\mathrm{R}}_\beta$
4	0.068	0	**4.40**	0	0	0	**8.71**	**0.54**	0	**2.95**	0	$\beta \in (\beta_4, \beta_3)$
3	36.3141	0	**4.09**	0	0	0	**8.88**	0	0	**3.01**	0	$\beta \in (\beta_3, \beta_1)$
2	4039	0	**3.33**	0	0	0	**9.17**	0	0	0	0	**no**
1	8011.68	0	0	0	0	0	**11.71**	0	0	0	0	$\beta \in (\beta_1, \beta_0)$
0	71166	0	0	0	0	0	0	0	0	0	0	$\beta > \beta_0$

Noisy data 2. The noise is nearly normal, centered, i.i.d., SNR= 25.74 dB:

$$d = \begin{pmatrix} 66.67 & -169.08 & 101.56 & 149.38 & -39.50 \end{pmatrix}^\mathrm{T} . \tag{54}$$

The optimal values $\{\mathrm{c}_k\}$ in Table 5 show that $\mathsf{L} = \mathsf{M}$. The sequence $\{\beta_\mathrm{k}\}$ by Definition 3 reads as

$$\beta_0 = 60287 \quad \beta_1 = 3825 \quad \beta_2 = 3037.1 \quad \beta_3 = 72.734 \quad \beta_4 = 0.0259 \quad \beta_5 = 0 \ . \tag{55}$$

This $\{\beta_\mathrm{k}\}$ is strictly decreasing and equals $\{\beta_\mathrm{k}\}$ in (40), as claimed in Lemma 8(a). From Theorem 9, problems $(\mathcal{C}_\mathrm{k})$ and $(\mathcal{R}_\beta)$ are quasi-completely equivalent. This is confirmed by the tests reported in Table 5.

Table 5: The optimal values and solutions of $(\mathcal{C}_k)$ for $k \in \mathbb{I}_4$ where d is given in (54). Here $\{\beta_k\}$ is strictly decreasing, see (55), so $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$ are quasi-completely equivalent.

k	c_k	$\widehat{C}_k$ = the optimal solution of $(\mathcal{C}_k)$, singleton										$\widehat{C}_k = \widehat{R}_\beta$
4	0.0259	0	**8.54**	0	0	**4.59**	**4.90**	**2.73**	0	0	0	$\beta \in (\beta_4, \beta_3)$
3	72.7601	0	**3.93**	0	0	0	**8.70**	0	0	**2.63**	0	$\beta \in (\beta_3, \beta_2)$
2	3109.86	0	**3.27**	0	0	0	**8.95**	0	0	0	0	$\beta \in (\beta_2, \beta_1)$
1	6934.85	0	0	0	0	0	**11.44**	0	0	0	0	$\beta \in (\beta_1, \beta_0)$
0	67222	0	0	0	0	0	0	0	0	0	0	$\beta > \beta_0$

8 Conclusions and future directions

We have derived the precise mechanism of the relationship between the global minimizers of least-squares constrained by k-sparsity (problem $(\mathcal{C}_k)$ in (2)) and regularized by $\|\cdot\|_0$ via a parameter $\beta > 0$ (problem $(\mathcal{R}_\beta)$ in (3)). Subsection 1.2 (Main contributions) made several claims regarding the obtained new results. Let us summarize.

(a) The constant L is the smallest number such that the optimal value of problem $(\mathcal{C}_\mathsf{L})$ is null; this L depends on the data but generically $\mathsf{L} = \mathsf{M}$. We have shown that for any $k \leqslant \mathsf{L}$, any global minimizer of $(\mathcal{C}_k)$ is isolated and has exactly k non-zero entries. When β ranges on $(0, +\infty)$, problem $(\mathcal{R}_\beta)$ has at most $\mathsf{L}+1$ different sets of global minimizers which are global minimizers of problems $(\mathcal{C}_k)$ for $k \in \{0, \ldots, \mathsf{L}\}$.

(b) Using the optimal values of problem $(\mathcal{C}_k)$ we proposed a sequence of critical parameters $\{\beta_k\}_{k=0}^{\mathsf{L}}$ with $\beta_\mathsf{L} = 0$ in Definition 3. Its largest strictly decreasing subsequence containing β_0 and having the smallest indexes $J \subset \{0, \ldots, \mathsf{L}\}$ is denoted by $\{\beta_{J_k}\}_{J_k \in J}$. For any $J_k \in J$ the global minimizers of problem $(\mathcal{C}_{J_k})$ and problem $(\mathcal{R}_\beta)$ for $\beta \in \left(\beta_{J_k}, \beta_{J_{k-1}}\right)$ coincide. For the isolated values β_{J_k}, the global minimizers of problem $\left(\mathcal{R}_{\beta_{J_k}}\right)$ contains those of problems $(\mathcal{C}_{J_k})$ and $\left(\mathcal{C}_{J_{k+1}}\right)$.

(c) When the whole sequence $\{\beta_k\}_{k=0}^{\mathsf{L}}$ in Definition 3 is strictly decreasing, it follows that problem $(\mathcal{C}_k)$ and problem $(\mathcal{R}_\beta)$ for all $\beta \in (\beta_k, \beta_{k-1})$ have the same set of global minimizers, for any $k \in \{0, \ldots, \mathsf{L}\}$. Then the sparsity constrained problem $(\mathcal{C}_k)$ and the regularized problem $(\mathcal{R}_\beta)$ are quasi-completely equivalent.

(d) The global minimizers of $(\mathcal{C}_k)$ and of $(\mathcal{R}_\beta)$ are generically unique.

(e) The Monte-Carlo tests (subsection 7.1) have shown that the degree of partial equivalence, i.e., the length of the effective critical sequence $\{\beta_k\}$, depends on the distribution of the coefficients of the matrix A and the data d.

The agreement between the optimal sets of problems $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$ is driven by the critical parameters $\{\beta_k\}_{k=0}^{\mathsf{L}}$ which depend on the data d via the optimal values of problem $(\mathcal{C}_k)$. The cases when β takes a critical value $\beta_k \in (0, +\infty)$ can be ignored – there are at most L such values (Remark 7). This, together with (d), tells us that the optimal sets of problems $(\mathcal{C}_k)$ and $(\mathcal{R}_\beta)$ are singletons (except for highly improbable data which can be ignored in practice).

Our theoretical findings pose intriguing questions, of both a theoretical and practical flavor.

- If one can solve problem $(\mathcal{C}_{\mathrm{k}})$ for all sparsity levels k, claims (b) and (c) shows that one will immediately deduce all possible global minimizers of problem $(\mathcal{R}_\beta)$. This suggests that a unifying framework for both problems might be developed.
- The results in (a) can clarify a proper choice between models $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ in applications. If one needs optimal solutions with a fixed number of nonzero entries, $(\mathcal{C}_{\mathrm{k}})$ is obviously the best choice. If only information on the perturbations is available, $(\mathcal{R}_\beta)$ is a more flexible model.
- Many algorithms have been built on good knowledge on the optimal solutions. One can expect our detailed results to give rise to innovative and efficient algorithms using (b) and (c).
- In the numerical tests, the quasi-complete equivalence scenario (c) was encountered in more than 93 % of the tests. This percentage may depend on the size of the tests, on the scaling of (A, d) and certainly on the distribution of the coefficients of (A, d). The question deserves a deeper exploration.
- By specifying a class of matrices A and assumptions on data d, one might want to infer statistical knowledge on the optimal values c_{k}'s of problems $(\mathcal{C}_{\mathrm{k}})$ and thus on the critical parameters $\{\beta_{\mathrm{k}}\}$. In the partial equivalence context (b), it would be intriguing to see if problem $(\mathcal{R}_\beta)$ is able to eliminate some uninteresting optimal solutions of problem $(\mathcal{C}_{\mathrm{k}})$. If the quasi-complete equivalence case (c) has appeared to be the main regime, there would be important practical challenges.

  Such a research direction is promising and various theoretical and practical results could be expected.
- Finally, we mention some useful extensions of our results.
  - Considering matrices A and data d with complex entries should not present inherent difficulties; however this is important in many applications (e.g., tomography, phase retrieval).
  - Extensions to penalties of the form $\|Du\|_0$ for D a linear operator are important in many applications (structured sparsity, imaging). However, preliminary research on the minimizers of the adaptations of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ to these more complex penalties must be conducted.

Readers may be interested in the relationship between the *local* minimizers of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$. Notice that every global minimizer is also a local minimizer.

Remark 11. From [30], *for any support* $\omega \subset \{0, 1, \ldots, \mathsf{M}\}$ the least squares solution $\widehat{u}_\omega$ of $\|A_\omega v - d\|^2$, completed with $\widehat{u}_{\mathrm{k}} = 0$ for all $\mathrm{k} \notin \omega$ (i.e., the solution of problem $(\mathcal{P}_\omega)$ in (9)), is *always* a local minimizer of problem $(\mathcal{R}_\beta)$, independently of the value of β. Following the equivalent formulation of problem $(\mathcal{C}_{\mathrm{k}})$ given in (10) and (11), any such $\widehat{u}$ is also a local minimizer of problem $(\mathcal{C}_{\mathrm{k}})$ for $\mathrm{k} = \sharp\,\mathrm{supp}(\widehat{u}) \leqslant \sharp\,\omega$. Enumerating all supports $\omega \subset \{0, 1, \ldots, \mathsf{M}\}$ shows that problem $(\mathcal{R}_\beta)$ (for some $\beta > 0$) and the family of problems $(\mathcal{C}_{\mathrm{k}})$ for $\mathrm{k} \in \{0, 1, \ldots, \mathsf{M}\}$ have the same sets of local minimizers. $\diamondsuit$

A similar statement on the strict local minimizers of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$ can also be obtained.

From (b) and this remark, it is easy to deduce that when the index set J of the parameters $\{\beta_{\mathrm{J}_{\mathrm{k}}}\}_{\mathrm{J}_{\mathrm{k}} \in \mathrm{J}}$ is strictly smaller than L, all optimal sets of problem $(\mathcal{C}_{\mathrm{k}})$ that cannot be global minimizers of problem $(\mathcal{R}_\beta)$ for any $\beta > 0$, are strict *local* minimizers of $(\mathcal{R}_\beta)$.

Remark 11 leads to new research directions. A crucial point is that the regularization parameter β will not play any role for the theoretical comparison between the sets of the local minimizers of problems $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$. Useful information on the number of *local* (and not global) minimizers of these problems could be inferred. Other important results would concern algorithms that are known to converge to local minimizers.

A Proofs for joint optimality conditions for $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$, section 2

A.1 On the optimal solutions of problem $(\mathcal{C}_{\mathrm{k}})$, subsection 2.2

Proof of Lemma 1. Using $(\mathcal{P}_\omega)$ in (9) and (11), the optimal value of $(\mathcal{C}_{\mathrm{k}})$ for any k is given by

$$\mathrm{c_k} = \inf\left\{ \|A\widetilde{u} - d\|^2 \text{ where } \widetilde{u} \in \mathbb{R}^{\mathsf{N}} \text{ solves } (\mathcal{P}_\omega) \;\Big|\; \omega \in \Sigma_{\mathrm{k}} \right\}.$$

For $\mathrm{k} \in \mathbb{I}_{\mathsf{N}}^0$ and $\omega \in \Sigma_{\mathrm{k}}$, define $\mathrm{c}^\omega \geqslant 0$ by

$$\mathrm{c}^\omega := \|A\widetilde{u} - d\|^2 \quad \text{where} \quad \widetilde{u} \quad \text{solves} \quad (\mathcal{P}_\omega) \quad \text{for} \quad \omega \in \Sigma_{\mathrm{k}} \ . \tag{56}$$

The set of numbers $\{\mathrm{c}^\omega \mid \omega \in \Sigma_{\mathrm{k}}\}$ is nonempty and finite. Then $\mathrm{c_k} = \min\{\mathrm{c}^\omega \mid \omega \in \Sigma_{\mathrm{k}}\}$ is well defined. By (56) there exists $\widehat{u} \in \mathbb{R}^{\mathsf{N}}$ such that $\|A\widehat{u} - d\|^2 = \mathrm{c_k}$. Hence $\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}$ and thus $\widehat{\mathrm{C}}_{\mathrm{k}} \neq \varnothing$.

Proof of Lemma 2 Since $\Sigma_{\mathrm{k-n}} \subset \Sigma_{\mathrm{k}}$, $\forall\, \mathrm{n} \in \mathbb{I}_{\mathrm{k}}^0$ it follows from (9) and (13) that

$$\mathrm{c_k} \leqslant \|Au - d\|^2 \quad \forall\, u \in \mathbb{R}^{\mathsf{N}} \ \text{ such that } \ \mathrm{supp}(u) \in \Sigma_{\mathrm{k-n}}, \quad \forall\, \mathrm{n} \in \mathbb{I}_{\mathrm{k}}^0 \ . \tag{57}$$

Hence $\mathrm{c_k} \leqslant \mathrm{c_{k-n}}$, $\forall\, \mathrm{n} \in \mathbb{I}_{\mathrm{k}}$. By H1, there is $\omega \in \Sigma_{\mathsf{M}}$ so that $\mathrm{rank}\,(A_\omega) = \mathsf{M} = \sharp\,\omega$. Then $\|A\widehat{u} - d\|^2 = \mathrm{c}_{\mathsf{M}} = 0$ for $\widehat{u}$ given by $\widehat{u}_\omega = (A_\omega)^{-1} d$ and $\widehat{u}_{\mathbb{I}_{\mathsf{N}} \setminus \omega} = 0$.

Proof of Lemma 3 Since $\mathrm{n} \geqslant 1$, $\widehat{u}$ solves the problem $\min_u \|Au - d\|^2$ subject to $\|\widehat{u}\|_0 < \mathrm{k}$. This is an unconstrained problem, therefore the gradient of $u \mapsto \|Au - d\|^2$ must be null at $\widehat{u}$:

$$A^{\mathrm{T}}(A\widehat{u} - d) = 0 \ .$$

By H1 we immediately get that $A\widehat{u} - d = A_{\widehat{\sigma}}\widehat{u}_{\widehat{\sigma}} - d = 0$ and $\|A\widehat{u} - d\|^2 = 0$. This combined with $\mathrm{supp}(\widehat{u}) \in \Sigma_{\mathrm{k-n}}$ yields $\mathrm{c_{k-n}} = 0$ and $\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k-n}}$. For any $\mathrm{m} \geqslant \mathrm{n} - \mathrm{k}$ one has $\mathrm{c_m} = 0$ by Lemma 2 and $\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{m}}$ because $\Sigma_{\mathrm{m}} \supset \Sigma_{\mathrm{k-n}}$.

In order to prove Theorem 1(a) we need some auxiliary results.

Corollary 3. *Let H1 hold. Then* $\left[\ \mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0 \quad \text{and} \quad \widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}} \quad \Longrightarrow \quad \|\widehat{u}\|_0 = \mathrm{k}\ \right]$.

Proof. The case $\mathrm{k} = 0$ being trivial we focus on $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}$. Assume that $\|\widehat{u}\|_0 = \mathrm{k} - \mathrm{n}$ for $\mathrm{n} \geqslant 1$. Then $\mathrm{c_{k-n}} = 0$ by Lemma 3 which contradicts the definition of L in (15) Hence $\mathrm{n} = 0$. □

Lemma 9. *Let H1 hold. Consider that* $\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}$ *for* $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0$. *Set* $\widehat{\sigma} := \mathrm{supp}(\widehat{u})$. *Then*

$$\mathrm{rank}\,(A_{\widehat{\sigma}}) = \sharp\,\widehat{\sigma} \equiv \|\widehat{u}\|_0 \ . \tag{58}$$

Proof. One has $\|\widehat{u}\|_0 = \mathrm{k}$ by Corollary 3. For $\mathrm{k} = 0$ (58) is obvious. Suppose that (58) fails for $\mathrm{k} \geqslant 1$:

$$\mathrm{rank}\,(A_{\widehat{\sigma}}) \leqslant \sharp\,\widehat{\sigma} - 1 \ . \tag{59}$$

The rank-nullity theorem [25] entails that $\dim\ker(A_{\widehat{\sigma}}) = \sharp\,\widehat{\sigma} - \mathrm{rank}\,(A_{\widehat{\sigma}}) \geqslant 1$. Take an arbitrary $v_{\widehat{\sigma}} \in \ker(A_{\widehat{\sigma}}) \setminus \{0\}$, set $v_{\mathbb{I}_{\mathsf{N}} \setminus \widehat{\sigma}} := 0$ and select an $i \in \widehat{\sigma}$ in order to define $\widetilde{u}$ by

$$\widetilde{u} := \widehat{u} - \widehat{u}_i \frac{v}{v_i} \ .$$

Then $\widetilde{u}_i = 0$ and $\widehat{u}_i \neq 0$, so $\widetilde{\sigma} := \mathrm{supp}\,(\widetilde{u}) \subsetneqq \widehat{\sigma}$, which leads to

$$\|\widetilde{u}\|_0 = \mathrm{k} - \mathrm{n} \quad \text{for} \quad \mathrm{n} := \|\widehat{u}\|_0 - \|\widetilde{u}\|_0 \geqslant 1 \ . \tag{60}$$

From $v_{\widehat{\sigma}} \dfrac{\widehat{u}_i}{v_i} \in \ker(A_{\widehat{\sigma}})$ one has $A\widehat{u} = A_{\widehat{\sigma}}\widehat{u}_{\widehat{\sigma}} = A_{\widehat{\sigma}}\left(\widehat{u}_{\widehat{\sigma}} - v_{\widehat{\sigma}}\dfrac{\widehat{u}_i}{v_i}\right) = A_{\widehat{\sigma}}\widetilde{u}_{\widehat{\sigma}} = A_{\widetilde{\sigma}}\widetilde{u}_{\widetilde{\sigma}} = A\widetilde{u}$. Then

$$\mathrm{c}_\mathrm{k} = \|A\widehat{u} - d\|^2 = \|A\widetilde{u} - d\|^2 \ . \tag{61}$$

This, together with the fact $\mathrm{supp}(\widetilde{u}) \in \Sigma_\mathrm{k}$ shows that $\widetilde{u} \in \widehat{\mathrm{C}}_\mathrm{k}$. Thus $\widetilde{u} \in \widehat{\mathrm{C}}_\mathrm{k}$ and $\|\widetilde{u}\|_0 \leqslant \mathrm{k} - 1$ by (60), hence $\mathrm{c}_{\mathrm{k}-1} = 0$ by Lemma 3, in contradiction to the definition of L in (15). So (59) fails. $\square$

Lemma 10. *Let H1 hold and let $\widehat{u}$ be a solution of problem $(\mathcal{C}_\mathrm{k})$ such that* $\mathrm{rank}\,(A_{\widehat{\sigma}}) = \sharp\,\widehat{\sigma}$ *for* $\widehat{\sigma} :=$ $\mathrm{supp}(\widehat{u})$. *Then $\widehat{u}$ is a strict minimizer of problem $(\mathcal{C}_\mathrm{k})$.*

Proof. Observe that $\widehat{u}$ solves the problem $\min\left\{\|Au - d\|^2 \mid u \in S\right\}$ where $S = \left\{v \in \mathbb{R}^\mathsf{N} \ : \ \mathrm{supp}(v) = \widehat{\sigma}\right\}$. The case $\widehat{u}$ being trivial, we focus on $\widehat{u} \neq 0$. Define $B := \left\{v \in \mathbb{R}^\mathsf{N} \mid \|v\|_\infty < \min_{i\in\widehat{\sigma}} |\widehat{u}_i|\right\}$ which is nonempty. Noticing that $\widehat{u} \in S$, one has

$$\widehat{u} + v \in S \qquad \forall\, v \in S \cap B \ . \tag{62}$$

Since $\mathrm{rank}\,(A_{\widehat{\sigma}}) = \sharp\,\widehat{\sigma}$, the mapping $\widehat{u}_{\widehat{\sigma}} \mapsto \|A_{\widehat{\sigma}}\widehat{u}_{\widehat{\sigma}} - d\|^2$ is strictly convex and has a unique solution $\widehat{u}_{\widehat{\sigma}}$. This, combined with (62) shows that $\|A(\widehat{u} + v) - d\|^2 > \|A\widehat{u} - d\|^2$ for any $v \in S \cap B \setminus \{0\}$. Hence $\widehat{u}$ is a strict minimizer of problem $(\mathcal{C}_\mathrm{k})$ (Definition 2). $\square$

Proof of Theorem 1. (a) Let $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}$ for $\mathrm{k} \in \mathbb{I}^0_\mathsf{L}$. By Corollary 3 and Lemma 9, $\mathrm{rank}\,(A_{\widehat{\sigma}}) = \mathrm{k} = \|\widehat{u}\|_0$ where $\widehat{\sigma} := \mathrm{supp}(\widehat{u})$, which proves (16). Since $\widehat{u}$ solves problem $(\mathcal{P}_{\widehat{\sigma}})$ in (9) where $\mathrm{rank}\,(A_{\widehat{\sigma}}) = \sharp\,\widehat{\sigma}$ it follows by Lemma 10 that $\widehat{u}$ is a strict minimizer.
(b) From Lemma 3 and the definition of L in (15) one finds $\widehat{\mathrm{C}}_\mathsf{L} \subset \widehat{\mathrm{C}}_\mathrm{k}$ for any $\mathrm{k} \geqslant \mathsf{L} + 1$.

Proof of Corollary 1 Let $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}$ and $\overline{u} \in \widehat{\mathrm{C}}_\mathrm{n}$ for $(\mathrm{k}, \mathrm{n}) \in (\,\mathbb{I}^0_\mathsf{L}\,)^2$, $\mathrm{k} \neq \mathrm{n}$. By Theorem 1(a), $\|\widehat{u}\|_0 = \mathrm{k}$ and $\|\overline{u}\|_0 = \mathrm{n}$, hence the result.

A.2 Proofs for necessary and sufficient conditions, subsection 2.3

From Theorems 2 and 3 all global minimizers $\widehat{u}$ of $\mathcal{F}_\beta$ satisfy $\|\widehat{u}\|_0 \leqslant \mathsf{M}$.

Corollary 4, Lemmas 11, 12 and 13 below help to prove Proposition 1.

Corollary 4 ([30] Corollary 3.3)**.** *Let $\widehat{u}$ solve $(\mathcal{P}_\omega)$ for $\omega \in \Omega_\mathrm{k}$ where $\mathrm{k} \in \mathbb{I}^0_\mathsf{M}$. Then $\widehat{u}$ is a strict (local) minimizer of $\mathcal{F}_\beta$ for any $\beta > 0$.*

With the notation in (17), it is suitable to set

$$\Omega := \bigcup_{\mathrm{k}=0}^{\mathsf{M}} \Omega_\mathrm{k} \ .$$

Lemma 11. *Let $d \in \mathbb{R}^\mathsf{M}$ and $\beta > 0$. Then*

$$\widehat{u} \text{ is a strict (local) minimizer of } \mathcal{F}_\beta \quad \Longleftrightarrow \quad \widehat{u} \in \mathrm{U} := \bigcup_{\omega\in\Omega} \{\widetilde{u} \in \mathbb{R}^\mathsf{N} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega\}. \tag{63}$$

Proof. Let $\widehat{u}$ be a strict (local) minimizers of $\mathcal{F}_\beta$. Then $\widehat{u}$ solves $(\mathcal{P}_\omega)$ for $\omega := \mathrm{supp}(\widehat{u})$. By Theorem 3 $\omega \in \Omega$ and thus $\widehat{u} \in \mathrm{U}$. Conversely, any $\widehat{u} \in \mathrm{U}$ is a strict (local) minimizer of $\mathcal{F}_\beta$ by Corollary 4. □

Now we partition U in (63) as follows:

$$\mathrm{U} = \bigcup_{\mathrm{k}=0}^{\mathsf{M}} \mathrm{U}_\mathrm{k} \quad \text{where} \quad \mathrm{U}_\mathrm{k} := \bigcup_{\omega \in \Omega} \left\{ \widetilde{u} \in \mathbb{R}^\mathsf{N} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega \text{ and } \|\widetilde{u}\|_0 = \mathrm{k} \right\} . \tag{64}$$

Lemma 12. *Let H1 be satisfied and* L *be as in* (15)*. Then* $\widehat{\mathrm{C}}_\mathrm{k} \subset \mathrm{U}_\mathrm{k} \quad \forall\, \mathrm{k} \in \mathbb{I}_\mathsf{L}^0$.

Proof. Let $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}$ for $\mathrm{k} \in \mathbb{I}_\mathsf{L}^0$. Set $\omega := \mathrm{supp}(\widehat{u})$. The expression for $\widehat{\mathrm{C}}_\mathrm{k}$ in (19) and Theorem 1(a) show that $\widehat{u}$ solves $(\mathcal{P}_\omega)$ for $\omega \in \Omega_\mathrm{k} \subset \Omega$ and that $\|\widehat{u}\|_0 = \mathrm{k}$. Hence $\widehat{u} \in \mathrm{U}_\mathrm{k}$. □

Lemma 13. *Let H1 hold,* L *be as in* (15) *and let* $\beta > 0$.

(a) *Let* $\mathrm{k} \in \mathbb{I}_\mathsf{L}^0$. *Then*

$$\mathcal{F}_\beta(\widehat{u}) = \mathrm{c}_\mathrm{k} + \beta\,\mathrm{k} \qquad \forall\, \widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}\ ; \tag{65}$$

$$\mathcal{F}_\beta(\widetilde{u}) > \mathcal{F}_\beta(\widehat{u}) \qquad \forall\, \widetilde{u} \in \mathrm{U}_\mathrm{k} \setminus \widehat{\mathrm{C}}_\mathrm{k}\ . \tag{66}$$

(b) *Let* $\widehat{u} \in \widehat{\mathrm{C}}_\mathsf{L}$. *If* $\mathsf{L} \leqslant \mathsf{M} - 1$*, then*

$$\mathcal{F}_\beta(\widetilde{u}) > \mathrm{c}_\mathsf{L} + \beta\,\mathsf{L} = \mathcal{F}_\beta(\widehat{u}) \qquad \forall\, \widetilde{u} \in \mathrm{U}_\mathrm{n} \quad \forall\, \mathrm{n} \in \{\mathsf{L}+1, \cdots, \mathsf{M}\}\ ; \tag{67}$$

and thus any $\widetilde{u} \in \mathrm{U}_\mathrm{n}$ *for* $\mathrm{n} \in \{\mathsf{L}+1, \cdots, \mathsf{M}\}$ *obeys* $\widetilde{u} \notin \widehat{\mathrm{R}}_\beta$ *for any* $\beta > 0$.

Proof. From the definition of U_k in (64), if $\mathrm{U}_\mathrm{k} \neq \varnothing$, then $\|\widetilde{u}\|_0 = \mathrm{k}$ for any $\widetilde{u} \in \mathrm{U}_\mathrm{k}$.

(a) Since $\mathrm{k} \in \mathbb{I}_\mathsf{L}^0$, $\widehat{\mathrm{C}}_\mathrm{k} \subset \mathrm{U}_\mathrm{k}$ by Lemma 12. Any $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}$ yields $\|A\widehat{u} - d\|^2 = \mathrm{c}_\mathrm{k}$, hence (65). Any $\widetilde{u} \in \mathrm{U}_\mathrm{k} \setminus \widehat{\mathrm{C}}_\mathrm{k}$ is not an optimal solution of $(\mathcal{C}_\mathrm{k})$, so $\|A\widetilde{u} - d\|^2 > \mathrm{c}_\mathrm{k}$. Then $\mathcal{F}_\beta(\widetilde{u}) = \|A\widetilde{u} - d\|^2 + \beta\,\mathrm{k} > \mathrm{c}_\mathrm{k} + \beta\,\mathrm{k} = \mathcal{F}_\beta(\widehat{u})$.

(b) By the definition of L, $\mathrm{c}_\mathrm{n} = \mathrm{c}_\mathsf{L} = 0$, $\forall\, \mathrm{n} \geqslant \mathsf{L}$. It follows that for any $\widetilde{u} \in \mathrm{U}_\mathrm{n}$, $\forall\, \mathrm{n} \geqslant \mathsf{L} + 1$ one has $\mathcal{F}_\beta(\widetilde{u}) = \|A\widetilde{u} - d\|^2 + \beta \mathrm{n} > \mathrm{c}_\mathsf{L} + \beta\mathsf{L} = \beta\mathsf{L}$. Such a $\widetilde{u}$ is not a global minimizer of $\mathcal{F}_\beta$. □

Proof of Proposition 1. (a) follows from Lemma 13(a).
(b)-(c) Let $\widehat{u} \in \widehat{\mathrm{R}}_\beta$. Then $\widehat{u}$ is a strict minimizer of $\mathcal{F}_\beta$ (Theorem 2(b)) and $\widehat{u} \in \mathrm{U}$ by Lemma 11. Set $\mathrm{k} := \|\widehat{u}\|_0$; then $\widehat{u} \in \mathrm{U}_\mathrm{k}$ according to (64). In addition, $\mathrm{k} \leqslant \mathsf{L}$ because otherwise $\widehat{u} \notin \widehat{\mathrm{R}}_\beta$ by Lemma 13(b). Also, $\widehat{u} \in \widehat{\mathrm{R}}_\beta$ means that $\mathcal{F}_\beta(\widehat{u})$ is the optimal value of problem $(\mathcal{R}_\beta)$. Then $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}$ by Lemma 13(a). Further, $\widehat{u} \mapsto \mathcal{F}_\beta(\widehat{u})$ is constant for any $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}$; see (65). Therefore, $\widehat{\mathrm{C}}_\mathrm{k} \subseteq \widehat{\mathrm{R}}_\beta$.

Proof of Lemma 4. The backward implication is obvious. We focus on the forward one. Let $\mathrm{k} \in \mathbb{I}_\mathsf{L}^0$; we proceed by contraposition. Assume that $\widehat{\mathrm{C}}_\mathrm{k} \cap \widehat{\mathrm{R}}_\beta \neq \varnothing$, i.e., there exists $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k} \cap \widehat{\mathrm{R}}_\beta$. Then $\widehat{\mathrm{C}}_\mathrm{k} \subseteq \widehat{\mathrm{R}}_\beta$ by Proposition 1(c), a contradiction to the fact that $\widehat{\mathrm{C}}_\mathrm{k} \not\subseteq \widehat{\mathrm{R}}_\beta$.

Proof of Theorem 4. By Proposition 1(b), any $\widehat{u} \in \widehat{\mathrm{R}}_\beta$ satisfies $\widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k}$ for $\mathrm{k} \leqslant \mathsf{L}$ and thus $\widehat{u} \in \widehat{\mathrm{C}}$. Therefore, $\widehat{\mathrm{R}}_\beta \subset \widehat{\mathrm{C}}$. The same holds for any $\beta > 0$ which proves the theorem.

Proof of Lemma 5 By Remark 5, $\widehat{\mathrm{C}}_\mathrm{k} \subseteq \widehat{\mathrm{R}}_\beta$ implies $\quad \mathcal{F}_\beta(\widehat{u}) \leqslant \mathcal{F}_\beta(\overline{u}) \quad \forall\, \widehat{u} \in \widehat{\mathrm{C}}_\mathrm{k} \quad \forall\, \overline{u} \in \widehat{\mathrm{C}}_{\mathrm{k+p}}$. Using Proposition 1(a) this inequality reads as $\quad \mathrm{c}_\mathrm{k} + \beta\,\mathrm{k} \leqslant \mathrm{c}_{\mathrm{k+p}} + \beta\,(\mathrm{k} + \mathrm{p})$, $\quad$ which leads to

$$\beta \geqslant \frac{\mathrm{c}_\mathrm{k} - \mathrm{c}_{\mathrm{k+p}}}{\mathrm{p}} \ . \tag{68}$$

On the other hand, $\widehat{\mathrm{C}}_{\mathrm{k+p}} = \widehat{\mathrm{R}}_{\beta'}$ entails $\quad \mathcal{F}_{\beta'}(\overline{u}) < \mathcal{F}_{\beta'}(\widehat{u}) \quad \forall\, \overline{u} \in \widehat{\mathrm{C}}_{\mathrm{k+p}} \quad \forall\, \widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}$. Therefore

$$\mathrm{c}_{\mathrm{k+p}} + \beta' \,(\mathrm{k}+\mathrm{p}) < \mathrm{c}_{\mathrm{k}} + \beta' \,\mathrm{k} \quad \Rightarrow \quad \beta' < \frac{\mathrm{c}_{\mathrm{k}} - \mathrm{c}_{\mathrm{k+p}}}{\mathrm{p}} \ . \tag{69}$$

Comparing (69) and (68) proves the first part of the lemma. The proof of second one is similar.

B Proofs for parameter values for equality between optimal sets, section 3

B.1 Proofs for the entire list of critical parameters values, subsection 3.1

Proof of Proposition 2 For $\mathrm{k} \in \{0, \mathsf{L}\}$ one has $\beta_{\mathrm{k}} < \beta_{\mathrm{k}}^{\mathrm{U}}$ by Remark 6. We recall the notation Ω_{k} introduced in (17) and that Π_ω is the orthogonal projector onto range(A_ω) given in Remark 4.

From Definition 3, there exists $\mathrm{n} \in \mathbb{I}_{\mathsf{L}-\mathrm{k}}$ such that $\beta_{\mathrm{k}} = \dfrac{\mathrm{c}_{\mathrm{k}} - \mathrm{c}_{\mathrm{k+n}}}{\mathrm{n}}$. By Theorem 1, there are $\omega \in \Omega_{\mathrm{k}}$ and $\overline{\omega} \in \Omega_{\mathrm{k+n}}$ obeying $\mathrm{c}_{\mathrm{k}} = d^{\mathrm{T}}(I - \Pi_\omega)\, d$ and $\mathrm{c}_{\mathrm{k+n}} = d^{\mathrm{T}}(I - \Pi_{\overline{\omega}})\, d$. Similarly, there is $\mathrm{m} \in \mathbb{I}_{\mathrm{k}}$ satisfying $\beta_{\mathrm{k}}^{\mathrm{U}} = \dfrac{\mathrm{c}_{\mathrm{k-m}} - \mathrm{c}_{\mathrm{k}}}{\mathrm{m}}$ and $\widehat{\omega} \in \Omega_{\mathrm{k-m}}$ such that $\mathrm{c}_{\mathrm{k-m}} = d^{\mathrm{T}}(I - \Pi_{\widehat{\omega}})\, d$. Then $\beta_{\mathrm{k}} - \beta_{\mathrm{k}}^{\mathrm{U}}$ reads as

$$\begin{aligned}
\beta_{\mathrm{k}} - \beta_{\mathrm{k}}^{\mathrm{U}} &= \frac{d^{\mathrm{T}}(\Pi_{\overline{\omega}} - \Pi_\omega)d}{\mathrm{n}} - \frac{d^{\mathrm{T}}(\Pi_\omega - \Pi_{\widehat{\omega}})d}{\mathrm{m}} && (70)\\
&= \frac{d^{\mathrm{T}}\big(\mathrm{m}\Pi_{\overline{\omega}} + \mathrm{n}\Pi_{\widehat{\omega}} - (\mathrm{m}+\mathrm{n})\Pi_\omega\big)d}{\mathrm{n}\,\mathrm{m}} \ . && (71)
\end{aligned}$$

It follows that all $d \in \mathbb{R}^{\mathsf{M}}$ leading to $\beta_{\mathrm{k}} - \beta_{\mathrm{k}}^{\mathrm{U}} = 0$ for some $\mathrm{k} \in \mathbb{I}_{\mathsf{L}-1}$ belong to the set S given below:

$$\mathrm{S} := \bigcup_{\mathrm{k}=1}^{\mathsf{L}-1} \mathrm{S}_{\mathrm{k}} \quad \text{where} \quad \mathrm{S}_{\mathrm{k}} := \bigcup_{\mathrm{n}=1}^{\mathsf{L}-\mathrm{k}} \bigcup_{\mathrm{m}=1}^{\mathrm{k}} \bigcup_{\omega \in \Omega_{\mathrm{k}}} \bigcup_{\overline{\omega} \in \Omega_{\mathrm{k+n}}} \bigcup_{\widehat{\omega} \in \Omega_{\mathrm{k-m}}} \ker\big(\mathrm{m}\Pi_{\overline{\omega}} + \mathrm{n}\Pi_{\widehat{\omega}} - (\mathrm{n}+\mathrm{m})\Pi_\omega\big) \ . \tag{72}$$

Let $\mathrm{k} \in \mathbb{I}_{\mathsf{L}-1}$. Since $\mathrm{rank}\,(\Pi_{\overline{\omega}}) = \mathrm{k}+\mathrm{n}$, the SVD of $\Pi_{\overline{\omega}}$ yields an orthonormal matrix $Q \in \mathbb{R}^{\mathsf{M}\times\mathsf{M}}$ so that

$$Q^{\mathrm{T}}\Pi_{\overline{\omega}}Q = \begin{bmatrix} I_{\mathrm{k+n}} & \vdots & 0 \\ \cdots & & \cdots \\ 0 & \vdots & 0 \end{bmatrix} .$$

Observe that $Q^{\mathrm{T}}\Pi_{\overline{\omega}}Q$ is symmetric semi-positive definite. By setting

$$P := \frac{\mathrm{n}}{\mathrm{m}} Q^{\mathrm{T}}\Pi_{\widehat{\omega}}Q = \begin{bmatrix} P_0 & \vdots & P_1 \\ \cdots & & \cdots \\ P_1^{\mathrm{T}} & \vdots & P_2 \end{bmatrix}$$

where P_0 is the first $(\mathrm{k}+\mathrm{n}) \times (\mathrm{k}+\mathrm{n})$ principal minor of P, we have

$$\begin{aligned}
\mathrm{rank}(\mathrm{m}\Pi_{\overline{\omega}} + \mathrm{n}\Pi_{\widehat{\omega}}) &= \mathrm{rank}\left(\Pi_{\overline{\omega}} + \frac{\mathrm{n}}{\mathrm{m}}\Pi_{\widehat{\omega}}\right) = \mathrm{rank}\left(Q^{\mathrm{T}}\left(\Pi_{\overline{\omega}} + \frac{\mathrm{n}}{\mathrm{m}}\Pi_{\widehat{\omega}}\right)Q\right) \\
&= \mathrm{rank}\begin{bmatrix} I_{\mathrm{k+n}} + P_0 & \vdots & P_1 \\ \cdots & & \cdots \\ P_1^{\mathrm{T}} & \vdots & P_2 \end{bmatrix} \geqslant \mathrm{k}+\mathrm{n} \ . && (73)
\end{aligned}$$

The rank inequality above comes from the fact that P_0 is positive semidefinite and thus $I_{\mathrm{k+n}} + P_0$ is positive definite; see [24]. This, together with the facts that $\mathrm{rank}\,((\mathrm{n}+\mathrm{m})\,(\Pi_\omega)) = \mathrm{k}$ and that $\mathrm{n} \geqslant 1$ in (73) gives that $\mathrm{m}\Pi_{\overline{\omega}} + \mathrm{n}\Pi_{\widehat{\omega}} - (\mathrm{n}+\mathrm{m})\Pi_\omega \neq 0$; hence $\dim\big(\ker(\mathrm{m}\Pi_{\overline{\omega}} + \mathrm{n}\Pi_{\widehat{\omega}} - (\mathrm{n}+\mathrm{m})\Pi_\omega)\big) \leqslant \mathsf{M}-1$. Therefore, S is a finite union of vector subspaces of dimension $\leqslant \mathsf{M}-1$.

Finally, if $d \in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{S}$, then $\beta_{\mathrm{k}} - \beta_{\mathrm{k}}^{\mathrm{U}} \neq 0$ in (70) because the term in (71) in non-null.

B.2 Proofs for effective parameters values, subsection 3.3

Proof of Lemma 6 The definition of J and J^E in (29) shows that

$$\mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0 \quad \text{and} \quad \beta_{\mathrm{k}} > \beta_{\mathrm{k}}^{\mathrm{U}} \quad \Longleftrightarrow \quad \mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0 \setminus \{\mathrm{J} \cup \mathrm{J}^{\mathrm{E}}\} \ .$$

By Theorem 6(a), one has $\widehat{\mathrm{C}}_{\mathrm{k}} \not\subseteq \widehat{\mathrm{R}}_\beta$, $\forall\, \beta > 0$, if and only if $\mathrm{k} \in \mathbb{I}_{\mathsf{L}}^0 \setminus \{\mathrm{J} \cup \mathrm{J}^{\mathrm{E}}\}$. By Lemma 4, $\widehat{\mathrm{C}}_{\mathrm{k}} \not\subseteq \widehat{\mathrm{R}}_\beta$ means $\widehat{\mathrm{C}}_{\mathrm{k}} \cap \widehat{\mathrm{R}}_\beta = \varnothing$, $\forall\, \beta > 0$.

Proof of Lemma 7 Let $\mathrm{m} \in \mathrm{J}^{\mathrm{E}}$. Theorem 6(a) shows that $\widehat{\mathrm{C}}_{\mathrm{m}} \subset \widehat{\mathrm{R}}_{\beta_{\mathrm{m}}}$. By Theorem 6(b) and Proposition 3(a), $\widehat{\mathrm{C}}_{\mathrm{J_k}} = \widehat{\mathrm{R}}_\beta$ if and only if $\beta_{\mathrm{J_k}} < \beta < \beta_{\mathrm{J_{k-1}}}$. Assume that $\mathrm{m} < \mathrm{J_k}$. Then Lemma 5 shows that $\beta_{\mathrm{m}} > \beta$ for any $\beta \in \big(\beta_{\mathrm{J_k}}, \beta_{\mathrm{J_{k-1}}}\big)$. It follows that $\beta_{\mathrm{m}} \neq \beta_{\mathrm{J_k}}$. This, together with (35), yields

$$\{\mathrm{m} \in \mathrm{J}^{\mathrm{E}} \mid \mathrm{m} < \mathrm{J_k}\} \cap \mathrm{J}^{\mathrm{E}}_{\beta_{\mathrm{J_k}}} = \varnothing \ . \tag{74}$$

Using the same statements, $\widehat{\mathrm{C}}_{\mathrm{J_{k+1}}} = \widehat{\mathrm{R}}_\beta$ if and only if $\beta \in \big(\beta_{\mathrm{J_{k+1}}},\ \beta_{\mathrm{J_k}}\big)$. Consider that $\mathrm{m} > \mathrm{J_{k+1}}$. From Lemma 5, $\beta_{\mathrm{m}} < \beta$ for any $\beta \in \big(\beta_{\mathrm{J_{k+1}}},\ \beta_{\mathrm{J_k}}\big)$ and thus $\beta_{\mathrm{m}} \neq \beta_{\mathrm{J_k}}$. Hence

$$\{\mathrm{m} \in \mathrm{J}^{\mathrm{E}} \mid \mathrm{m} > \mathrm{J_{k+1}}\} \cap \mathrm{J}^{\mathrm{E}}_{\beta_{\mathrm{J_k}}} = \varnothing \ . \tag{75}$$

Jointly (74), (75) and $\mathrm{J} \cap \mathrm{J}^{\mathrm{E}} = \varnothing$ prove (36). Finally, $\beta_{\mathrm{k_p}} \equiv \beta_{\mathsf{L}} = 0$ in (31) shows that $\mathrm{J}^{\mathrm{E}}_{\beta_{\mathsf{L}}} = \varnothing$.

Proof of Proposition 4 From Definition 4 one has $0 \in \mathrm{J}$ and from Proposition 3(a) $\{\beta_{\mathrm{k}}\}_{\mathrm{k}\in\mathrm{J}}$ is strictly decreasing. By Lemma 7 it holds that $\mathrm{J_k} < \mathrm{m}$ for any m satisfying $\beta_{\mathrm{J_k}} = \beta_{\mathrm{m}}$. Hence J contains the smallest indexes.

If $\mathrm{J} = \mathbb{I}_{\mathsf{L}}^0$, the result is proved. Consider that $\mathrm{J} \subsetneqq \mathbb{I}_{\mathsf{L}}^0$. Suppose that there is $\mathrm{m} \in \mathbb{I}_{\mathsf{L}} \setminus \mathrm{J}$ such that $\{\beta_{\mathrm{k}}\}_{\mathrm{k}\in\mathrm{J}^*}$ is strictly decreasing where J^* denotes the increasingly ordered $\{\mathrm{J} \cup \{\mathrm{m}\}\}$. By Proposition 3(c), $\mathrm{m} \notin \mathrm{J}^{\mathrm{E}}$. Then $\mathrm{m} \in \mathbb{I}_{\mathsf{L}}^0 \setminus \{\mathrm{J} \cup \mathrm{J}^{\mathrm{E}}\}$. Since $\beta_{\mathrm{J_k}} < \beta_{\mathrm{J_{k-1}}}$, $\forall\, \mathrm{J_k} \in \mathrm{J}$, there are $(\mathrm{J_{k-1}}, \mathrm{J_k})$ such that $\mathrm{J_{k-1}} < \mathrm{m} < \mathrm{J_k}$ and $\beta_{\mathrm{J_k}} < \beta_{\mathrm{m}} < \beta_{\mathrm{J_{k-1}}}$. From Theorem 6(b) and Proposition 3(a), $\widehat{\mathrm{R}}_\beta = \widehat{\mathrm{C}}_{\mathrm{J_k}}$ if and only if $\beta \in \big(\beta_{\mathrm{J_k}},\ \beta_{\mathrm{J_{k-1}}}\big)$. Since $\beta_{\mathrm{m}} \in \big(\beta_{\mathrm{J_k}},\ \beta_{\mathrm{J_{k-1}}}\big)$ one has

$$\mathcal{F}_{\beta_{\mathrm{m}}}(\overline{u}) > \mathcal{F}_{\beta_{\mathrm{m}}}(\widehat{u}) \quad \forall\, \overline{u} \in \widehat{\mathrm{C}}_{\mathrm{m}} \quad \forall\, \widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{J_k}} \ ,$$

which yields $\mathrm{c_m} + \beta_{\mathrm{m}}\, \mathrm{m} > \mathrm{c_{J_k}} + \beta_{\mathrm{m}}\, \mathrm{J_k}$ (see Proposition 1(a)). Consequently,

$$\beta_{\mathrm{m}} < \frac{\mathrm{c_m} - \mathrm{c_{J_k}}}{\mathrm{J_k} - \mathrm{m}} \ .$$

However, using that $\mathrm{J_k} > \mathrm{m}$, Definition 3 shows that $\beta_{\mathrm{m}} \geqslant \dfrac{\mathrm{c_m} - \mathrm{c_{J_k}}}{\mathrm{J_k} - \mathrm{m}}$, a contradiction.

C Proofs for optimal values of $(\mathcal{C}_{\mathrm{k}})$ and $(\mathcal{R}_\beta)$, section 5

In order to prove Proposition 5 we shall use Lemmas 14 and 15 given below.

Lemma 14. *Let H1 hold and let* $\mathrm{k} \in \{1, \ldots, \mathsf{M} - 1\}$. *Then*

$$d \in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{G_k} \quad \Longleftrightarrow \quad \mathrm{c_k} > 0 \ .$$

Proof. Let $d \in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{G_k}$ and let $\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k}}$. Set $\widehat{\sigma} := \mathrm{supp}(\widehat{u})$. By Theorem 1(a) and the notation Ω_{k} in (17), $\widehat{\sigma} \in \Omega_{\mathrm{k}}$. Since $(I - \Pi_{\widehat{\sigma}})$ is the orthogonal projector onto $(\mathrm{range}(A_{\widehat{\sigma}}))^{\perp}$, one has

$$d \in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{G_k} \quad \Longrightarrow \quad d \not\in \mathrm{range}(A_{\widehat{\sigma}}) \;\text{ and }\; d \neq 0 \quad \Longrightarrow \quad \mathrm{c_k} = \|A\widehat{u} - d\|^2 = d^{\mathrm{T}}(I - \Pi_{\widehat{\sigma}})\, d > 0 \ .$$

Conversely, let $\mathrm{c_k} > 0$. If $d \in \mathrm{G_k}$, there is $\omega \in \Omega_{\mathrm{k}}$ meeting $d \in \mathrm{range}(A_\omega)$. For $u_\omega = \left(A_\omega^{\mathrm{T}} A_\omega\right)^{-1} A_\omega^{\mathrm{T}} d$ one has $\|A_\omega u_\omega - d\|^2 = d^{\mathrm{T}}(I - \Pi_\omega)\, d = 0$, a contradiction to $\mathrm{c_k} > 0$. □

Lemma 15. *Let H1 hold and let* $\mathrm{E_k}$ *be given by* (45). *For any* $\mathrm{k} \geqslant 1$ *such that* $\mathrm{c_{k-1}} > 0$ *one has*

$$\mathrm{c_{k-1}} > \mathrm{c_k} \qquad \forall\, d \in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{E_k} \ .$$

Proof. From H1, there is $\mathrm{n} \in \mathbb{I}_{\mathsf{N}}$ meeting $\langle A_{\mathrm{n}}, d \rangle \neq 0$. Set $\widehat{u}_{\mathrm{n}} = \arg\min_{v \in \mathbb{R}} \|A_{\mathrm{n}}\, v - d\|^2$. Then

$$\widehat{u}_{\mathrm{n}} = \frac{\langle A_{\mathrm{n}}, d \rangle}{\|A_{\mathrm{n}}\|^2} \neq 0 \quad \text{and} \quad \mathrm{c}_1 \leqslant \|A_{\mathrm{n}}\, \widehat{u}_{\mathrm{n}} - d\|^2 = \|d\|^2 - \frac{\langle A_{\mathrm{n}}, d \rangle^2}{\|A_{\mathrm{n}}\|^2} < \mathrm{c}_0 = \|d\|^2 \ .$$

Consider that $\mathrm{k} \geqslant 2$. Let $\widehat{u} \in \widehat{\mathrm{C}}_{\mathrm{k-1}}$ and $\mathrm{c_{k-1}} > 0$. Set $\widehat{\sigma} := \mathrm{supp}(\widehat{u})$ and denote by $B_{\widehat{\sigma}}$ a matrix whose columns form an orthonormal basis for $A_{\widehat{\sigma}}$. From Theorem 1(a), $\widehat{\sigma} \in \Omega_{\mathrm{k-1}}$. By H1, there is $\mathrm{n} \in \mathbb{I}_{\mathsf{N}} \setminus \widehat{\sigma}$ such that $\omega := \widehat{\sigma} \cup \{\mathrm{n}\} \in \Omega_{\mathrm{k}}$. Then there is $b_{\mathrm{k}} \in \mathrm{range}(A_\omega)$ such that $B_\omega = (B_{\widehat{\sigma}}\, ,\, b_{\mathrm{k}})$ forms an orthonormal basis for A_ω (Gram-Schmidt theorem, see, e.g., [20]). The orthogonal projectors onto $\mathrm{range}(A_{\widehat{\sigma}})$ and $\mathrm{range}(A_\omega)$ are $\Pi_{\widehat{\sigma}} = B_{\widehat{\sigma}} B_{\widehat{\sigma}}^{\mathrm{T}}$ and $\Pi_\omega = B_\omega B_\omega^{\mathrm{T}} = B_{\widehat{\sigma}} B_{\widehat{\sigma}}^{\mathrm{T}} + b_{\mathrm{k}} b_{\mathrm{k}}^{\mathrm{T}}$, respectively. Then $\mathrm{c_k} \leqslant \zeta_{\mathrm{k}} := d^{\mathrm{T}}(I - \Pi_\omega)\, d$. Applying (20) yields

$$\mathrm{c_{k-1}} - \mathrm{c_k} \geqslant \mathrm{c_{k-1}} - \zeta_{\mathrm{k}} = d^{\mathrm{T}} \left(\Pi_\omega - \Pi_{\widehat{\sigma}}\right) d \;=\; \langle b_{\mathrm{k}}, d \rangle^2 \ .$$

Since $d \in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{E_k}$, one has $d \not\in \big(\mathrm{range}\big(\,(B_{\widehat{\sigma}}\, ,\, b_{\mathrm{k}})\,\big)\big)^{\perp}$. Hence $\langle b_{\mathrm{k}}, d \rangle^2 > 0$. □

Proof of Proposition 5 (a) Let $d \in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{G}_{\mathsf{L}'-1}$. By Lemma 14, $\mathrm{c}_{\mathsf{L}'-1} > 0$. Since $\{\mathrm{c_k}\}$ is decreasing (Lemma 2), one has $\mathrm{c_k} > 0 \; \forall\, \mathrm{k} \leqslant \mathsf{L}' - 1$. Conversely, if $d \not\in \mathbb{R}^{\mathsf{M}} \setminus \mathrm{G}_{\mathsf{L}'-1}$, then $\mathrm{c}_{\mathsf{L}'-1} = 0$ by Lemma 14.

(b) Using (a)

$$d \in \mathbb{R}^{\mathsf{M}} \setminus \left(\mathrm{E}_2 \cup \mathrm{G}_{\mathsf{L}'-1}\right) \quad \Longrightarrow \quad \mathrm{c_k} > 0 \quad \forall\, \mathrm{k} \in \mathbb{I}^0_{\mathsf{L}'-1} \ . \tag{76}$$

Let $\mathrm{k} \in \mathbb{I}_{\mathsf{L}'}$. For any $\omega \in \Omega_{\mathrm{k}}$ and $\overline{\omega} \subsetneqq \omega$ with $\sharp\, \overline{\omega} = \mathrm{k} - 1$ one has $\overline{\omega} \in \Omega_{\mathrm{k-1}}$ and $\mathrm{range}(A_{\overline{\omega}}) \subsetneqq \mathrm{range}(A_\omega)$. Since $(\mathrm{range}(A_{\overline{\omega}}))^{\perp} \supsetneqq (\mathrm{range}(A_\omega))^{\perp}$ we obtain $\left(\mathbb{R}^{\mathsf{M}} \setminus \mathrm{E_k}\right) \supsetneqq \left(\mathbb{R}^{\mathsf{M}} \setminus \mathrm{E_{k-1}}\right)$ for any $\mathrm{k} \in \{2, \ldots, \mathsf{L}'\}$. Using Lemma 15 together with (76) shows that

$$d \in \mathbb{R}^{\mathsf{M}} \setminus \left(\mathrm{E}_2 \cup \mathrm{G}_{\mathsf{L}'-1}\right) \quad \Longrightarrow \quad \mathrm{c_{k-1}} > \mathrm{c_k} > 0 \quad \forall\, \mathrm{k} \in \mathbb{I}_{\mathsf{L}'-1} \quad \text{and} \quad \mathrm{c}_{\mathsf{L}'-1} > \mathrm{c}_{\mathsf{L}'} \ .$$

Proof of Corollary 2. (a) follows from Theorem 7 and Proposition 1(a).

(b) By (46), r_β is the lower envelope of $\mathsf{L} + 1$ affine increasing functions. Hence (b).

(c) One has $\beta_{\mathrm{J_k}}(\mathrm{J_{k+1}} - \mathrm{J_k}) = \mathrm{c_{J_k}} - \mathrm{c_{J_{k+1}}}$ and $\{\beta_{\mathrm{J_k}}\}_{\mathrm{k}=0}^{\mathrm{p}}$ strictly decreasing by Proposition 3. Then

$$\begin{aligned} r_{\beta_{\mathrm{J_k}}} - r_{\beta_{\mathrm{J_{k+1}}}} &= \mathrm{c_{J_k}} - \mathrm{c_{J_{k+1}}} + \beta_{\mathrm{J_k}} \mathrm{J_k} - \big(\beta_{\mathrm{J_k}} - \big(\beta_{\mathrm{J_k}} - \beta_{\mathrm{J_{k+1}}}\big)\big)\, \mathrm{J_{k+1}} \\ &= \mathrm{c_{J_k}} - \mathrm{c_{J_{k+1}}} + \beta_{\mathrm{J_k}}(\mathrm{J_k} - \mathrm{J_{k+1}}) + \big(\beta_{\mathrm{J_k}} - \beta_{\mathrm{J_{k+1}}}\big)\, \mathrm{J_{k+1}} = \big(\beta_{\mathrm{J_k}} - \beta_{\mathrm{J_{k+1}}}\big)\, \mathrm{J_{k+1}} > 0 \ . \end{aligned} \tag{77}$$

Since $\mathrm{J}_0 = 0$, see (31), $r_{\beta_{\mathrm{J}_0}} = \mathrm{c}_{\mathrm{J}_0} = r_\beta$, $\forall\, \beta \geqslant \beta_{\mathrm{J}_0}$. Using (a) and (77) yields $r_{\beta_{\mathrm{J}_0}} > r_\beta$, $\forall\, \beta < \beta_{\mathrm{J}_0}$.

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References

[1] H. Attouch, J. Bolte, and B. F. Svaiter, *Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forwardbackward splitting, and regularized Gauss-Seidel methods*, Math. Program., 137 (2013).

[2] A. Auslender and M. Teboulle, *Asymptotic Cones and Functions in Optimization and Variational Inequalities*, Springer, New York, 2003.

[3] E. Van Den Berg and M. P. Friedlander, *Sparse optimization with least-squares constraints*, SIAM J. Optim., 21 (2011), pp. 1201–1229.

[4] D. Blanchard, C. Cartis, J. Tanner, and A. Thompson, *Phase transitions for greedy sparse approximation algorithms*, Appl. Comput. Harmon. Anal., 30 (2011), pp. 188–203.

[5] T. Blumensath and M. Davies, *Iterative thresholding for sparse approximations*, J. Fourier Anal. Appl., 14 (2008), pp. 629–654.

[6] ——, *Iterative hard thresholding for compressed sensing*, Appl. Comput. Harmon. Anal., 27 (2009), pp. 265–274.

[7] ——, *Normalized iterative hard thresholding: Guaranteed stability and performance*, IEEE J. Sel. Topics Signal Process., 4 (2010), pp. 298–309.

[8] A. M. Bruckstein, D. L. Donoho, and M. Elad, *From sparse solutions of systems of equations to sparse modeling of signals and images*, SIAM Rev., 51 (2009), pp. 34–81.

[9] C. Cartis and A. Thompson, *A new and improved quantitative recovery analysis for iterative hard thresholding algorithms in compressed sensing*, IEEE Trans. Inform. Theory, 61 (2015), pp. 2019–2042.

[10] A. Cohen, W. Dahmien, and R. A. DeVore, *Compressed sensing and best k-term approximation*, J. Amer. Math. Soc., 22 (2009), pp. 211–231.

[11] R. A. DeVore, *Nonlinear approximation*, Acta Numerica, 7 (1998).

[12] B. Dong and Y. Zhang, *An efficient algorithm for ℓ_0 minimization in wavelet frame based image restoration*, J. Sci. Comput., 54 (2013).

[13] D.L. Donoho, *Compressed sensing*, IEEE Trans. Inform. Theory, 52 (2006), pp. 1289–1306.

[14] Y. C. Eldar and G. Kutyniok, *Compressed Sensing: Theory and Applications*, Cambridge Univ. Press, 2012.

[15] J. Fan and R. Li, *Statistical challenges with high dimensionality: feature selection in knowledge discovery*, in Proc. Intern. Congr. Math., vol. 3, Eur. Math. Soc., Zürich, 2006, pp. 595–622.

[16] M. Fornasier and R. Ward, *Iterative thresholding meets free-discontinuity problems*, Found. Comput. Math., 10 (2010), pp. 527–567.

[17] S. Geman and D. Geman, *Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images*, IEEE Trans. Pattern Anal. Mach. Intell., PAMI-6 (1984), pp. 721–741.

[18] J. Haupt and R. Nowak, *Signal reconstruction from noisy random projections*, IEEE Trans. Inform. Theory, 52 (2006), pp. 4036–4048.

[19] Y. Liu and Y. Wu, *Variable selection via a combination of the ℓ_0 and ℓ_1 penalties*, J. Comp. Graph. Stat., 16 (2007), pp. 782–798.

[20] D.G. Luenberger, *Optimization by Vector Space Methods*, Wiley, J., New York, 1st ed., 1969.

[21] J. Lv and Y. Fan, *A unified approach to model selection and sparse recovery using regularized least squares*, The Annals of Statistics, 37 (2009).

[22] A. Maleki and D. L. Donoho, *Optimally tuned iterative reconstruction algorithms for compressed sensing*, IEEE J. Sel. Topics Signal Process., 4 (2010), pp. 330–341.

[23] R. T. Marler and J. S. Arora, *Survey of multi-objective optimization methods for engineering*, Struct; Multidisc; Optim., 26 (2004), pp. 369–395

[24] C. D. Jr. Meyer, *Generalized inverses and ranks of block matrices*, SIAM J. Appl. Math., 25 (1973), pp. 597–602.

[25] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.

[26] A. J. Miller, *Subset Selection in Regression*, Chapman and Hall, London, U.K., 2nd ed., 2002.

[27] D. Needell, and J. Tropp, *CoSaMP: Iterative signal recovery from incomplete and inaccurate samples*, Appl. Comput. Harmon. Anal., 26 (3) (2009), pp. 301–321.

[28] J. Nocedal and S. Wright, *Numerical Optimization*, Springer, NY, 2 ed., 2006.

[29] J. Neumann, C. Schörr, and G. Steidl, *Combined SVM-based Feature Selection and classification*, Machine Learning 61, 2005, pp. 129–150.

[30] M. Nikolova, *Description of the minimizers of least squares regularized with ℓ_0-norm. Uniqueness of the global minimizer*, SIAM J. Imaging Sci., 6 (2013), pp. 904–937.

[31] M. C. Robini, A. Lachal, and I.E. Magnin, *A stochastic continuation approach to piecewise constant reconstruction*, IEEE Trans. Image Process., 16 (2007), pp. 2576–2589.

[32] M. C. Robini and I. E. Magnin, *Optimization by stochastic continuation*, SIAM J. Imaging Sci., 3 (2010), pp. 1096–1121.

[33] M. C. Robini and P.-J. Reissman, *From simulated annealing to stochastic continuation: a new trend in combinatorial optimization*, J. Global Optim., 56 (2013), pp. 185–215.

[34] R. T. Rockafellar and J. B. Wets, *Variational analysis*, Springer-Verlag, New York, 1998.

[35] E. Soubies, L. Blanc-Féraud and G. Aubert, *A Continuous Exact ℓ_0 penalty (CEL0) for least squares regularized problem*, SIAM J. Imaging Sci., 8 (2015), pp. 1607–1639.

[36] J.A. Tropp, *Just relax: convex programming methods for identifying sparse signals in noise*, IEEE Trans. Inform. Theory, 52 (2006), pp. 1030–1051.

[37] J. Tropp and S. J. Wright, *Computational methods for sparse solution of linear inverse problems*, Proc. IEEE, 98 (2010), pp. 948–958.