V Introduction to the Direct method in the calculus of variations

Questions: (how to ensure existence and uniqueness of a minimizer of J
how to do it the conditions fail . . . .

1. Topologies

$(X, |\cdot|)$ Banach space

(topological dual $X^* = \{ \ell : X \to \mathbb{R}$ linear

$$|\ell|_{X^*} = \sup \frac{|\ell(u)|}{|u|_X} < \infty \}$$

Two topologies on X

(a) strong $\quad u_n \xrightarrow[X]{} u \qquad |u_n - u|_X \to 0$ as $n \to \infty$

$\Downarrow$

(b) weak $\quad u_n \underset{X}{\rightharpoonup} u \qquad \ell(u_n) \xrightarrow[n\to\infty]{} \ell(u) \quad \forall \ell \in X^*$

Three topologies on X^*

(a) Strong $\quad \ell_n \to \ell \qquad |\ell_n - \ell|_{X^*} \to 0$

(b) weak $\quad \ell_n \underset{X^*}{\rightharpoonup} \ell \qquad z(\ell_n) \xrightarrow[n\to\infty]{} z(\ell) \quad \forall z \in X^{**}$

(c) weak* $\quad \ell_n \overset{*}{\rightharpoonup} \ell \qquad \ell_n(u) \xrightarrow[n\to\infty]{} \ell(u) \quad \forall u \in X$

Remind. X is reflexive if $X^{**} = X$

$L^1(\Omega)$ is non reflexive $\qquad L^p(\Omega)$ $1<p<\infty$ are reflexive

2. Main properties

Thm [weak sequential compactness] (WSC)

(i) X - reflexive Banach, $K>0$, $u_n \in X$ $\forall n \geq 0$ (integer) sequence such that $|u_n|_X \leq K$

$\boxed{\ell(u_{n_j}) \to \ell(u) \; \forall \ell \in X^*}$

$\Rightarrow \exists u \in X \quad \exists u_{n_j}$ such that $u_{n_j} \xrightarrow[X]{} u$

(ii) X separable Banach, $K>0$ $\ell_n \in X^*$ $\forall n \geq 0$ such that $|\ell_n|_{X^*} \leq K$

$\Rightarrow \exists \ell \in X^* \quad \exists \ell_{n_j}$ such that $\ell_{n_j} \xrightarrow[X^*]{*} \ell$

Rmk the weak * topology allows one to obtain compactness results even if X is not reflexive

3. Existence of a minimizer

X Banach. $\quad \inf_{u \in X} J(u)$

The usual steps to prove existence of a minimizer

(A) Construct a minimizing sequence $u_n \in X$, $n \in \mathbb{N}$ i.e. a sequence satisfying $\lim_{n \to \infty} J(u_n) = \inf_{u \in X} J(u)$

(B) If J is coercive we can obtain an uniform bound $|u_n|_X \leq C$

If X is reflexive, WSC thm :

$\exists \hat{u} \in X$ and $\exists u_{n_j}$: $u_{n_j} \underset{X}{\rightharpoonup} \hat{u}$.

(c) To prove that $\hat{u}$ is a minimizer it is sufficient to obtain the inequality

$$\liminf_{u_{n_j} \rightharpoonup \hat{u}} J(u_{n_j}) \geq J(\hat{u})$$ which implies $$J(\hat{u}) = \min_{u \in X} J(u)$$

weak lower semi continuity

<u>Def.</u> J is l.s.c. for the weak topology (weakly l.s.c) if for all $u_n \rightharpoonup u_0$ we have

$$\liminf_{u_n \rightharpoonup u_0} J(u_n) \geq J(u_0)$$

Rmk. it is difficult in general to prove weak l.s.c

if $F : X \to \mathbb{R}$ is convex then

F weakly lsc $\iff$ F strongly lsc.

strong lsc is not hard to prove.

let $\Omega \subset \mathbb{R}^n$ be open, bounded

$$\boxed{J(u) = \int_\Omega F(u, Du, x)\, dx} \qquad J$$

$F : \mathbb{R}^{m=1} \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ continuous such that

$$0 \leq F(u, \xi, x) \leq f(|u|, |\xi|, x)$$

where f is increasing wrt $|u|, |\xi|$ and is integrable wrt x.

Let $W^{1,p}(\Omega) = \{ u \in L^p(\Omega),\ Du \in L^p(\Omega) \}$

Sobolev space

distributional sense

We consider $\mathcal{J}$ for $u \in W^{1,p}(\Omega)$.

Thm $J(u)$ is (sequentially) weakly l.s.c. on $W^{1,p}(\Omega)$ for $1 < p < \infty$ (weakly* l.s.c. if $p = \infty$)

$\Longleftrightarrow$ F is convex in ξ

Remarks
- $\exists$ nonconvex pbs admitting a solution
- the Thm is true ALSO if $m > 1$ and $n = 1$

it is no longer true if $m > 1$ and $n > 1$

(then a weaker notion of convexity is introduced)

Suppose $u(x) \in \mathbb{R}$

Thm (Existence) $\Omega \subset \mathbb{R}^n$ bounded

$F : \mathbb{R} \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ continuous and satisfies

(a) $F(u, \xi, x) \geq f(x) + b\,|u|^p + c\,|\xi|^p$, $p > 1$

$\forall\,(u, \xi, x)$ where $f \in L^1(\Omega)$, $b > 0$, $c > 0$ constants

(coercivity condition $\Rightarrow$ boundedness of the minimizing sequences)

(b) $\xi \to F(u, \xi, x)$ convex $\forall (u, x)$

(permits to pass to the limit of these sequences)

(c) $\exists\, u_0 \in W^{1,p}(\Omega)$ such that $J(u_0) < \infty$ (the pb. has a meaning)

Then;

$$\inf \left\{ J(u) = \int_\Omega F(u, Du, x)\,dx \ ,\ u \in W^{1,p}(\Omega) \right\}$$

admits a solution.

Moreover: if $(u,\xi) \to F(u,\xi,x)$ is strictly convex $\forall x \in \Omega$ then the solution is unique.

Comments: convexity used to obtain weak lsc., Coercivity - related to w. compactness.

4. Bad examples

$\Omega =]0,1)$

(A) $n = m = 1$ $\qquad F(u,\xi,x) = x\xi^2$

$$M = \inf \left\{ \int_0^1 (u'(x))^2 x\,dx \quad \overbrace{u(0)=1 \quad u(1)=0}^{BC} \right\}$$
$$u \in W^{1,2}(\Omega)$$

F is convex but not coercive wrt u because $F=0$ if $x=0$.

$\boxed{M=0}$ - proof:

minimizing sequence

$$u_n(x) = \begin{cases} 1 & \text{if } x \in (0,\frac{1}{n}] \\ -\dfrac{\log(x)}{\log(n)} & \text{if } x \in [\frac{1}{n}, 1) \end{cases}$$

$\Rightarrow u_n \in W^{1,\infty}$

$$u_n'(x) = \begin{cases} 0 & \text{if } x \in (0,\frac{1}{n}] \\ -\dfrac{1}{x\log(n)} & \text{if } x \in [\frac{1}{n}, 1) \end{cases}$$

$$J(u_n) = \int_{\frac{1}{n}}^{1} \cancel{x} \frac{1}{x^{\cancel{2}}(\log(n))^2} dx = \frac{1}{(\log n)^2} \underbrace{\int_{\frac{1}{n}}^{1} \frac{1}{x} dx}$$

$$= \frac{1}{\log n} \xrightarrow[n\to 0]{} 0$$

$$[\log x]_{\frac{1}{n}}^{1} = 0 - \log \frac{1}{n} = \log(n)$$

$\mu = 0$.

If $\exists\, \hat{u}$ minimizer we should have

$F(\hat{u}) = 0 \iff \hat{u}' = 0$ a.e. on $(0,1)$

this is incompatible with the B.C.

(B) Minimal surfaces $F(u, \xi, x) = \sqrt{u^2 + \xi^2}$ — convex, coercive

on $W^{1,1}(\Omega)$ – non reflexive

$$\mu = \inf\left\{ \int_0^1 \sqrt{u^2 + u'^2}\, dx ,\ u \in W^{1,1}(\Omega),\ u(0)=0 \quad u(1) = 1 \right\}$$

one proves that $\mu = 1$. Indeed:

$$J(u) = \int_0^1 \sqrt{u^2 + u'^2}\, dx \geq \int_0^1 |u'|\, dx \geq \int_0^1 u'\, dx = 1$$

$$\Rightarrow \mu \geq 1$$

$$u_n(x) = \begin{cases} 0 & \text{if } x \in (0, 1 - \frac{1}{n}] \\ 1 + n(x-1) & \text{if } x \in (1 - \frac{1}{n}, 1) \end{cases}$$

$$u_n'(x) = \begin{cases} 0 & \text{if } x \in (0, 1 - \frac{1}{n}] \\ n & \text{if } x \in (1 - \frac{1}{n}, 1) \end{cases}$$

$$F(u) = \sqrt{u^2+u'^2} = \begin{cases} 0 & \text{if } x \in (0, 1-\frac{1}{n}) \\ \sqrt{1+n(nx^2+2x-2)} & \text{in } x \in (1-\frac{1}{n}, 1) \end{cases}$$

$J(u_n) \longrightarrow 1$ if $n \to \infty \implies \mu = 1$.

Assume $\exists \hat{u}$ minimizer then

$$1 = J(\hat{u}) = \int_0^1 \sqrt{\hat{u}^2 + \hat{u}'^2}\, dx \geq \int_0^1 |\hat{u}'|\, dx \geqq$$

$$\geq \int_0^1 \hat{u}'(x)\, dx = 1$$

There is no solution satisfying the B.C.

(c) Bolza. $F(u, \xi, x) = u^2 + (\xi^2 - 1)^2$

$$\inf \left\{ J(u) = \int_0^1 (u^2 + (1-u'^2)^2)\, dx \right.$$
$$\left. u \in W^{1,4}(0,1) \text{ and } u(0) = u(1) = 0 \right\}$$

The pb. here is that F is nonconvex wrt ξ.

Easy to see that $\inf J = 0$ but corresponds to $\hat{u} = 0$ then BC fail.

Once we have the existence of a minimum – write the optimality conditions (EL)

Conversely if J is convex then a solution of EL solves the minimization problem.

5. Relaxation

If F is not weakly lsc.

in general - no hope of obtaining existence of a min of J

important idea: We could associate with J another functional $\mathcal{R}J$ whose minima = weak cluster points of all minimizing sequences of J.

eg. for Bolza pb. - define the lower s.c. envelope of J

X - Banach $\quad J : X \to \overline{\mathbb{R}}$

denote by τ the topology of X (strong or weak)

Def. (relaxed functional)

The τ-lsc envelop (= relaxed func.) $\mathcal{R}_\tau J$ of J is defined $\forall u \in X$ by

$$\mathcal{R}_\tau J(u) = \sup\{ G(u) : G \in \Gamma \}$$

$$\Gamma = \{ G : X \to \mathbb{R} \ \tau\text{-l.s.c. such that } G(u) \leq J(u) \ \forall u \in X \}$$

Thm (characterize the relaxed func.)

(i) $\forall u_n \xrightarrow{(\tau)} u \in X \quad \mathcal{R}_\tau J(u) \leq \liminf\limits_{n\to\infty} J(u_n)$

(ii) $\forall u \in X \ \exists (u_n)$ τ-conv. to u in X such that

$\mathcal{R}_\tau J(u) \geq \limsup\limits_{n\to\infty} J(u_n)$

The relationship btw $\inf\{J(u) : u \in X\}$
and $\inf\{\mathcal{R}_\tau J(u) : u \in X\}$?

Thm (main properties)

X reflexive Banach; τ - The weak topo.

Assume $J : X \to \overline{\mathbb{R}}$ coercive. Then

(i) $\mathcal{R}_\tau J$ is coercive and weakly lsc

(ii) $\mathcal{R}_\tau J$ has a minimizer $\in X$

(iii) $\min_{u \in X} \mathcal{R}_\tau J(u) = \inf_{u \in X} J(u)$

(iv) $\forall$ cluster point of a minimizing sequence for J is a minimizer of $\mathcal{R}_\tau J$

(v) $\forall$ minimizer of $\mathcal{R}_\tau J$ is the limit of a minimizing sequence for J.

One possibility to construct $\mathcal{R}_\tau J$ is to take $\underline{F^{**} \text{ with respect to } \xi}$

$$\mathcal{R}_\tau J(u) = \int_\Omega \underset{\substack{\uparrow \\ \text{wrt } \xi \text{ (the gradient variable)}}}{F^{**}} (u, \nabla u, x)\, dx.$$